Production of dissipative vortices by solid bodies in incompressible flows: Prandtl, Navier-Stokes and Euler solutions

Marie Farge,

Ecole Normale Supérieure, Paris

Romain Nguyen van yen, Matthias Waidman and Rupert Klein

Freie Universität, Berlin

Kai Schneider,

Université d'Aix-Marseille

Ocean Scale Interactions, a tribute to Bach-Lien Hua

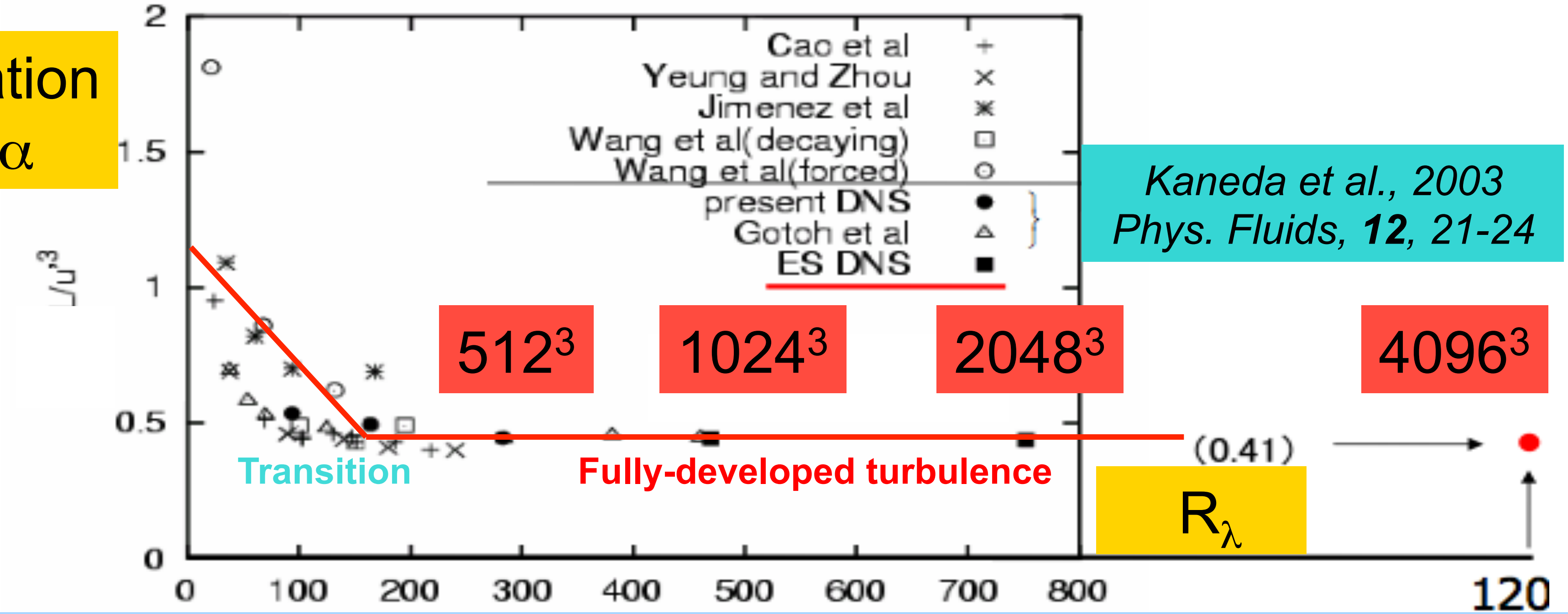
IFREMER, Brest, June 23rd 2014

Dissipation rate versus Reynolds

Normalized energy dissipation $\alpha \rightarrow$?
as $\nu \rightarrow 0$, or Re $\rightarrow \infty$

$$\alpha = \epsilon L / u'^3$$

Dissipation rate α



Kaneda et al., 2003
Phys. Fluids, ***12****, 21-24*

Both laboratory experiments and numerical experiments of turbulent flows show that the dissipation rate becomes independent of the fluid viscosity

What is the inviscid limit of Navier-Stokes?

Navier-Stokes equations with no-slip boundary conditions:

$$(\mathrm{NS})\quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p \boxed{+ \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\mathbf{u}} \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases}$$

?

Navier-Stokes solutions $\mathbf{u}_{\mathrm{Re}}(t,\mathbf{x})$ for $\nu \to 0$, $\mathrm{Re} \to +\infty$

The Reynolds number Re = U $/\nu$ appears when non dimensional quantities are introduced.

?

Euler equations with slip b.c.:

$$(\mathrm{E})\quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p \\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \mathbf{u}_{|\partial\Omega}\cdot\mathbf{n} = \mathbf{0}, \qquad \mathbf{u}(0,\cdot) = \mathbf{v} \end{cases}$$

?

Euler's solutions $\mathbf{u}(t,\mathbf{x})$ for $\nu = 0$, $\mathrm{Re} = +\infty$

Well posedness of Navier-Stokes and Euler

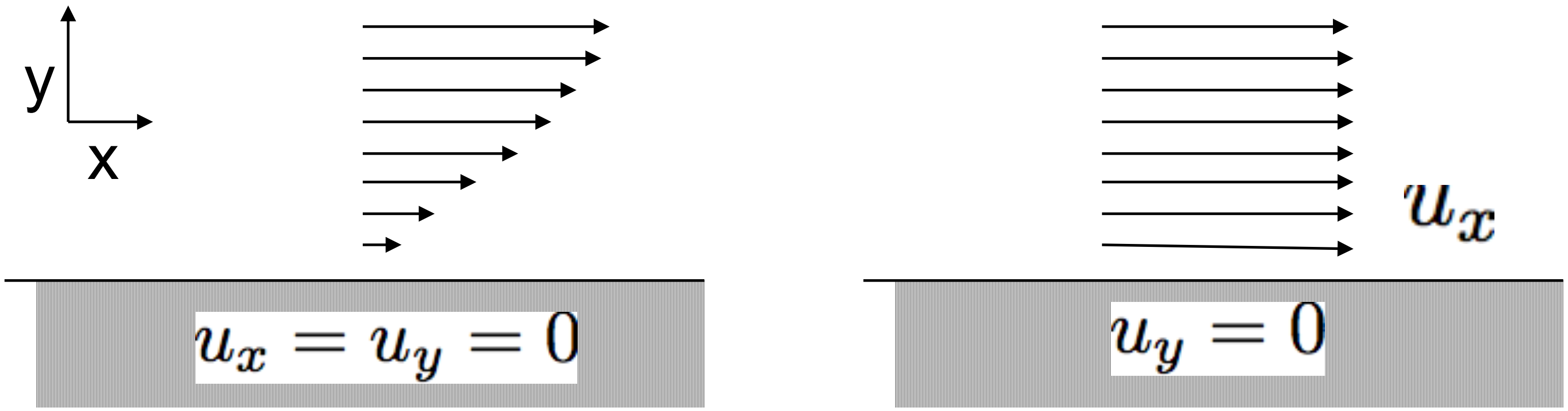
- *In 2D open space (without wall),*
  - for smooth initial data, Euler and Navier-Stokes equations are well posed (long time existence and uniqueness),
  - the Navier-Stokes equation is well posed in L^2 (energy norm),
  - the Euler equation is well posed for bounded vorticity,
  - for Euler equation, many open questions for cases with unbounded vorticity.

- *In 3D open space (without wall),*
  - for smooth initial data, both problems are well posed, at least for a short time,
  - the Navier-Stokes equation admits a weak solution for all time, but uniqueness is an open question,
  - for Euler equation even existence is an issue for long times.

- *In 2D and 3D confined space (with walls),*

the problem is still fully open for Euler and Navier-Stokes!

What is the problem with walls ?



No-slip b.c. **Slip b.c.**

- The wall imposes a strong tangential constraint on Navier-Stokes viscous flows,
- No boundary condition affects the tangential velocity for Euler inviscid flows.
- Navier's b. c. (1822) : $$u_x + \alpha \partial_y u_x = 0$$

u_x slip velocity α slip length $\partial_y u_x$ wall shear

Dissipation of energy in the inviscid limit

What happens for $\nu \to 0$?

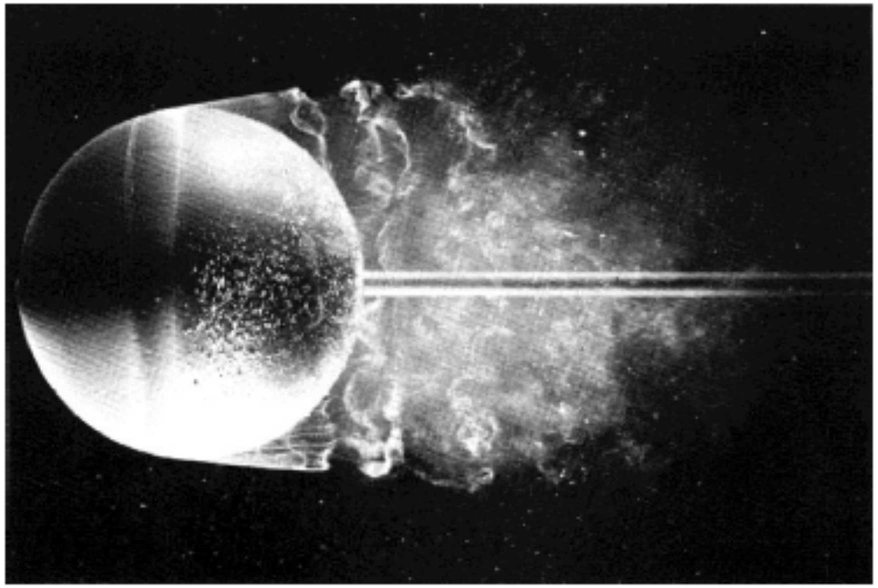
- In an incompressible flow ($\rho = 1$)

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{\mathbf{u}^2}{2} = -\nu \int \omega^2 = -2\nu Z$$

- To dissipate energy, vorticity needs to be created and/or amplified, in such a way that $Z \sim \nu^{-1}$.

Possible vorticity distributions:
$\omega \sim \nu^{-1/2}$ over $O(1)$ area,
$\omega \sim \nu^{-1}$ over $O(\nu)$ area.

E energy, Z enstrophy,
ν fluid kinematic viscosity
ω flow vorticity.



Why is dissipation of energy so essential ?

- Kato (1984) proved (roughly stated):

The NS solution converges towards the Euler solution in L^2:

$$\forall t \in [0,T], \left\| u_{\mathrm{Re}}(t) - u(t) \right\|_{L^2(\Omega)} \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0,$$

if and and only if

the energy dissipation during this interval vanishes:

$$\Delta E_{\mathrm{Re}}(0,T) = \mathrm{Re}^{-1} \int_0^T \mathrm{d}t \int_\Omega \mathrm{d}\mathbf{x} \left| \nabla \mathbf{u}(t,\mathbf{x}) \right|^2 \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0,$$

and even if and only if

it vanishes in a strip of width prop to **Re^{-1}** around the solid:

$$\mathrm{Re}^{-1} \int_0^T \mathrm{d}t \int_{\Gamma_{c\,\mathrm{Re}^{-1}}} \mathrm{d}\mathbf{x} \left| \nabla \mathbf{u}(t,\mathbf{x}) \right|^2 \underset{\mathrm{Re}\to\infty}{\longrightarrow} 0, \qquad \Gamma_{c\,\mathrm{Re}^{-1}} = \left\{ \mathbf{x} \,\middle|\, d(\mathbf{x},\partial\Omega) < c\,\mathrm{Re}^{-1} \right\}.$$

An important practical consequence

- To have any chance of observing energy dissipation (i.e. default of convergence towards the Euler solution), we need a smaller grid than Prandtl's (1904) prediction for attached boundary layers:

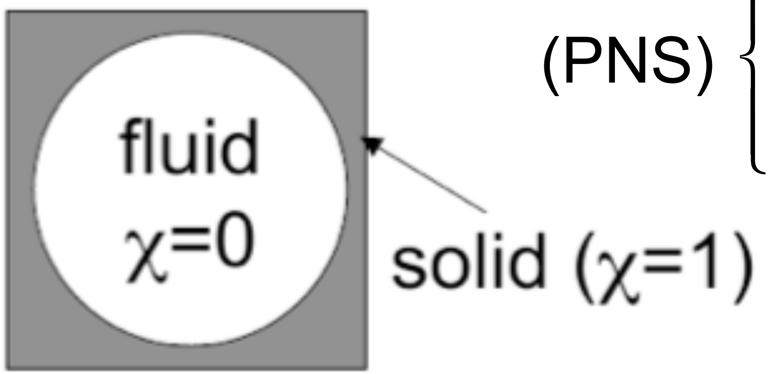
$$\cancel{\delta x \propto \mathrm{Re}^{-\frac{1}{2}}}$$

$$\downarrow$$

$$\boxed{\delta x \propto \mathrm{Re}^{-1}}$$

Volume penalization method

- For efficiency and simplicity, we would like to stick to a spectral solver in periodic, cartesian coordinates.
- as a counterpart, we need to add an additional term in the equations to approximate the effect of the boundaries,
- the geometry is encoded in a mask function χ ,



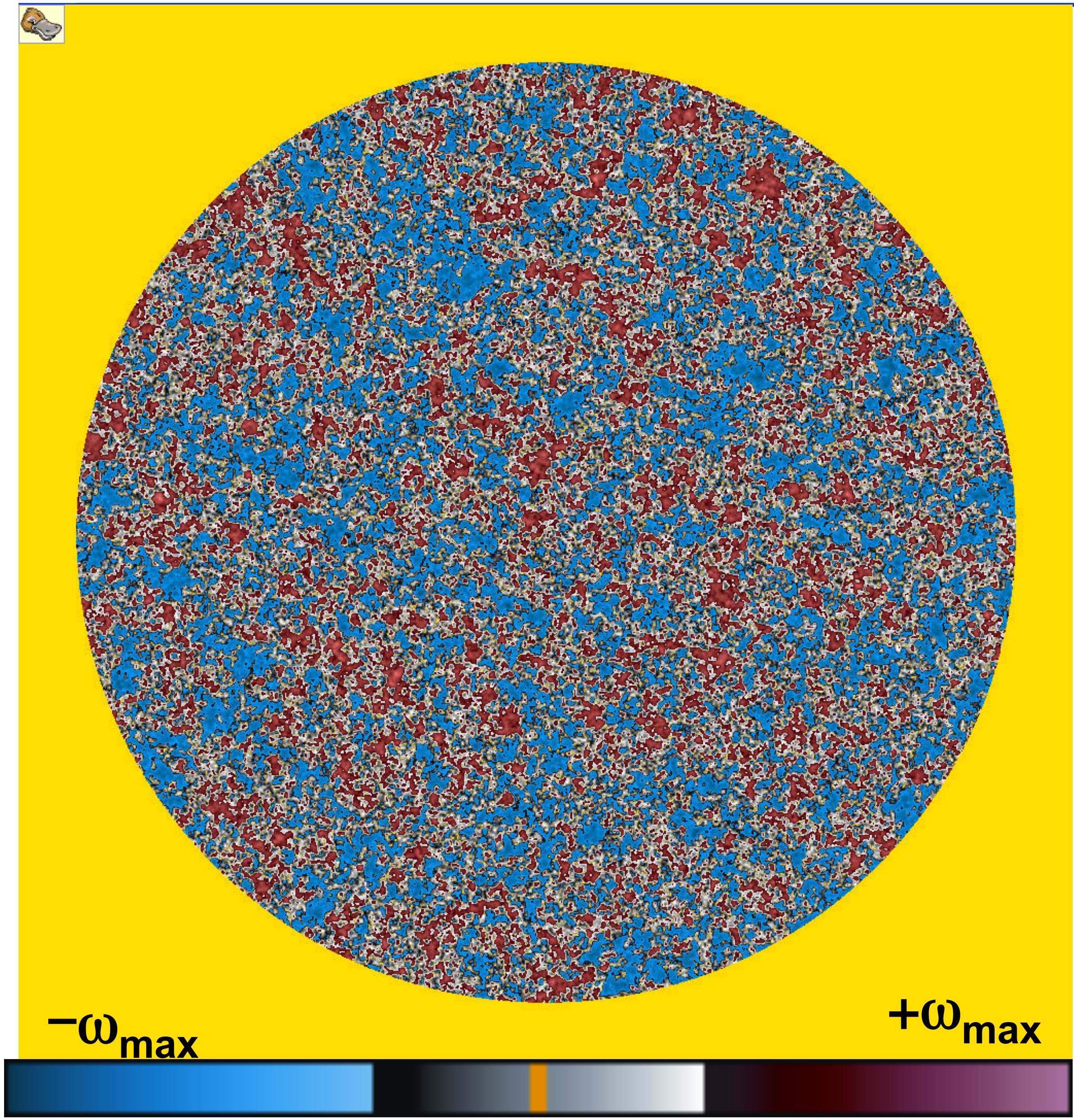
$$\text{(PNS)}\quad \begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\nabla p + \dfrac{1}{\mathrm{Re}}\Delta\mathbf{u} - \dfrac{1}{\eta}\chi_0\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}(0,\mathbf{x}) = \mathbf{v} \end{cases}$$

$$\xrightarrow{\text{solution}} \quad \mathbf{u}_{\mathrm{Re},\eta}$$

E. Arquis and J.P.Caltagirone,CRAS, 1984

*M. F. and K. Schneider, PRL, **95**, 2005*

Wall-bounded 2D turbulent flow

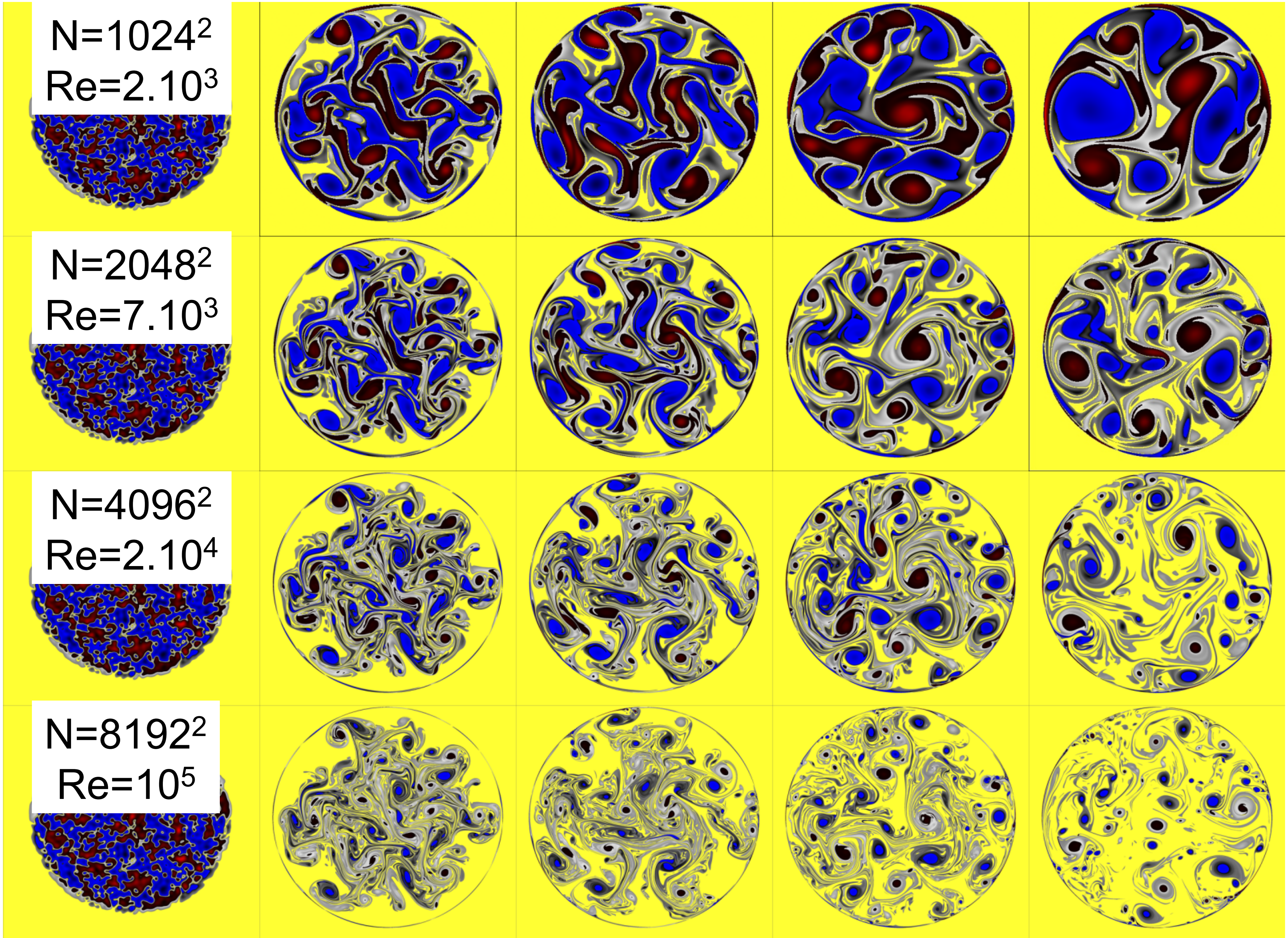


DNS
Resolution
$N=1024^2$

Random initial conditions

Pseudo-spectral method with volume penalization

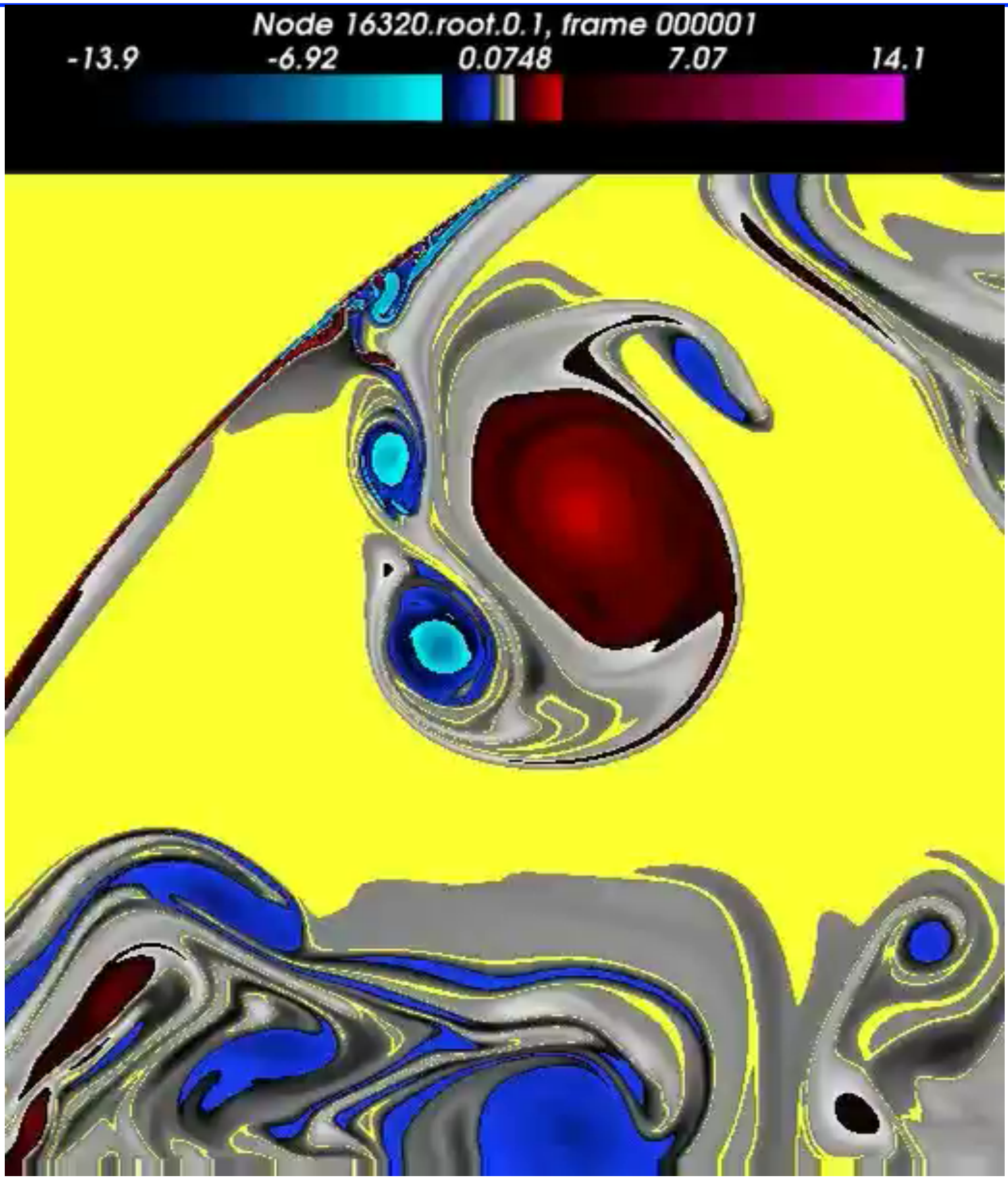
K. Schneider and M. F., Phys. Rev. Lett., ***95****, 244502 (2005)*



N=1024²
Re=2.10³
N=2048²
Re=7.10³
N=4096²
Re=2.10⁴
N=8192²
Re=10⁵

DNS of 2D confined flow

Resolution
$N=8192^2$



Time evolution of vorticity at the wall computed on IBM Blue-Gene, IDRIS, 2010

(100 Tflops)

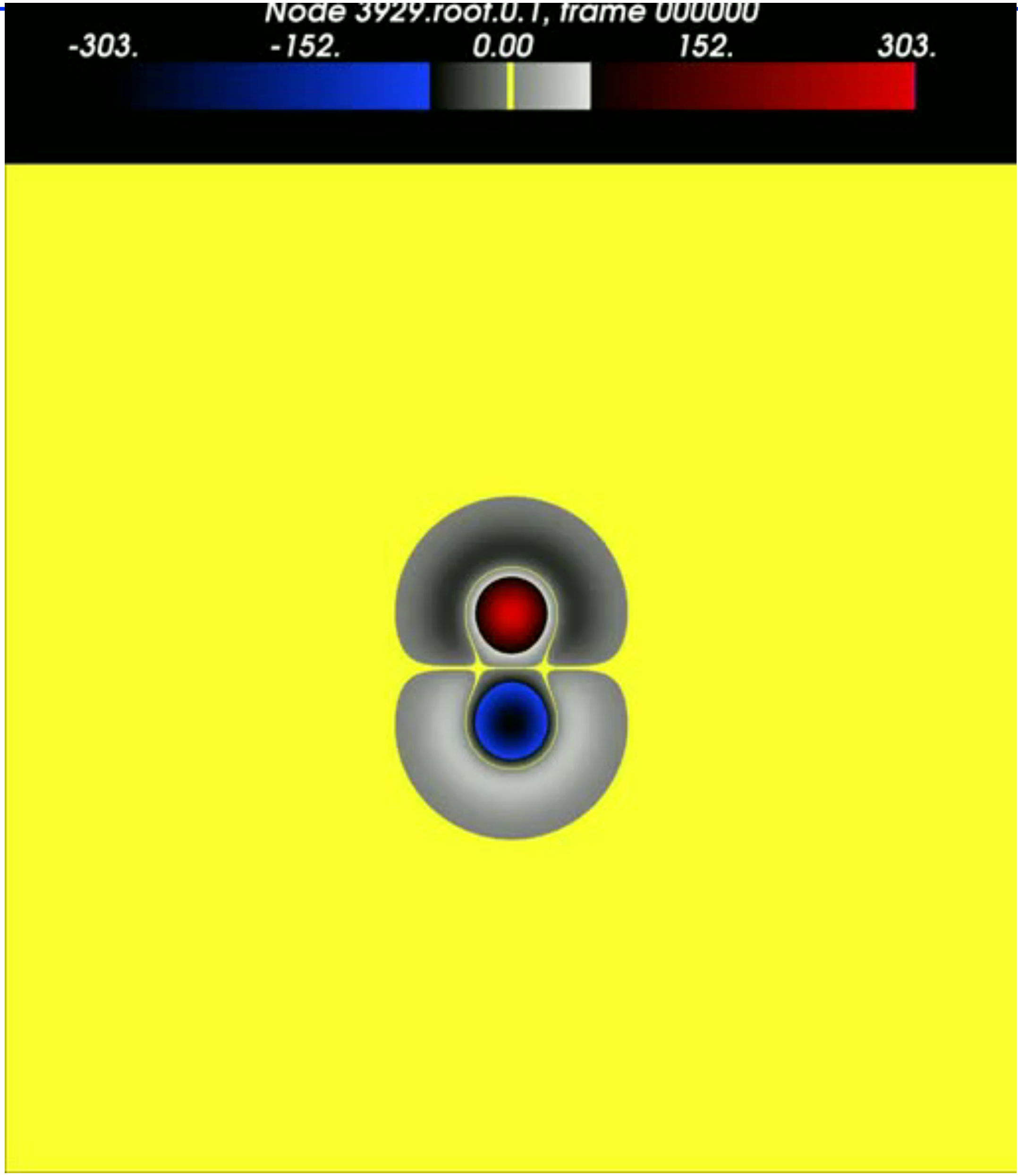
Nguyen van yen, M. F. and Schneider, 2010

Dipole impinging on a wall at Re= 2500

M. M. Koochesfahni and C. P. Gendrich
Michigan State University

Dipole-wall collision at Re=8000

DNS
Resolution
$N=8192^2$



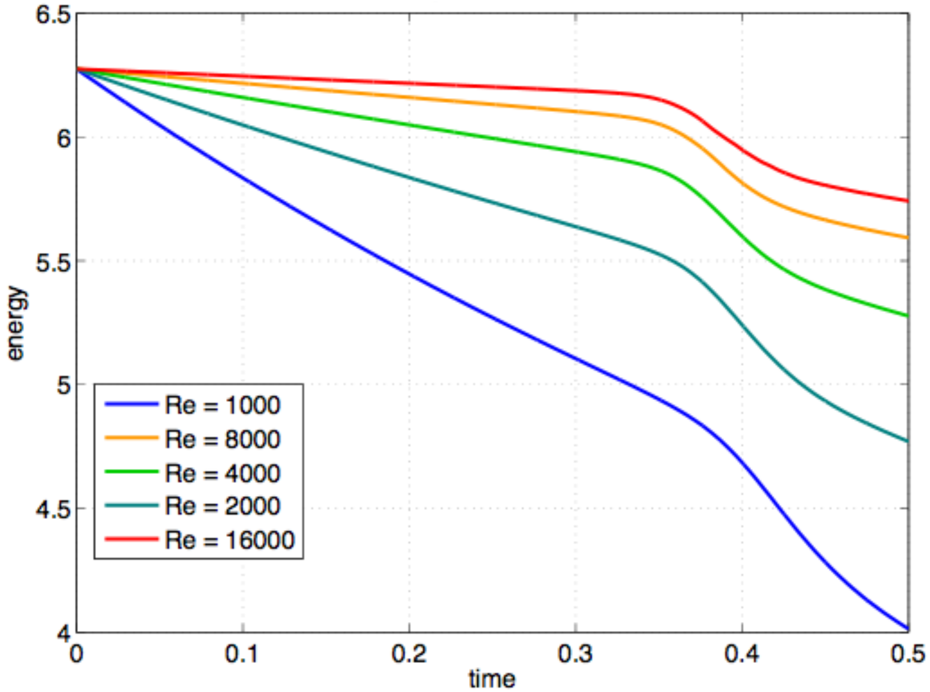
Navier-Stokes

$\nu>0$

Dipole-wall collision

DNS
Resolution
N=8192^2

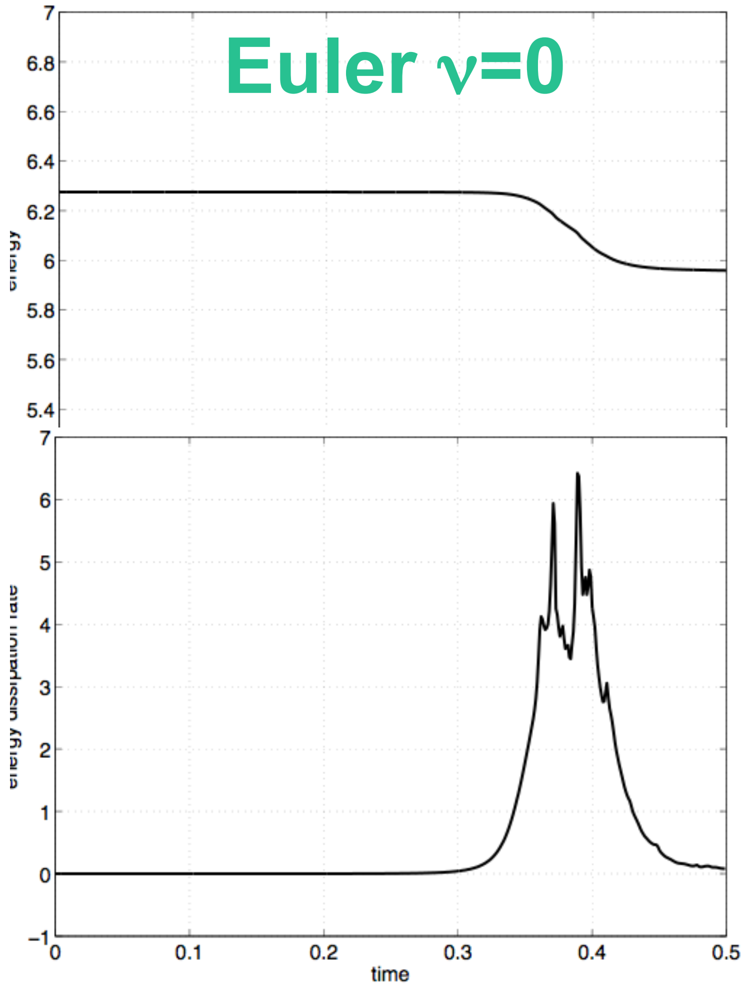
Navier-Stokes $\nu>0$



Time evolution of energy

Time evolution of energy

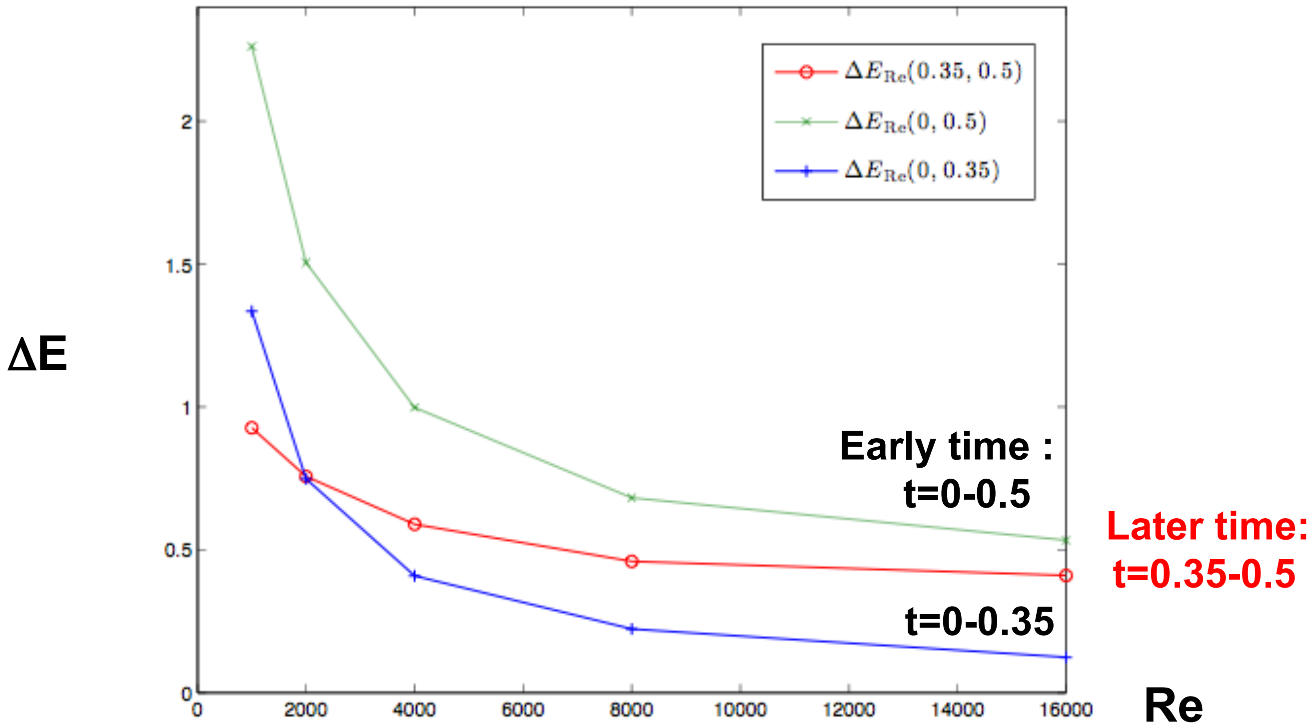
Euler $\nu=0$



and of energy dissipation rate

Energy dissipation

Energy dissipated
during the dipole-wall collision for
increasing Reynolds numbers



What are dissipative structures ?

- Our experiments with the dipole-wall collision suggest that the flow remains dissipative in the inviscid limit,
- it is tempting to relate these structures to energy dissipation,
- the kinetic energy density $e = \dfrac{|\mathbf{u}|^2}{2}$ obeys:

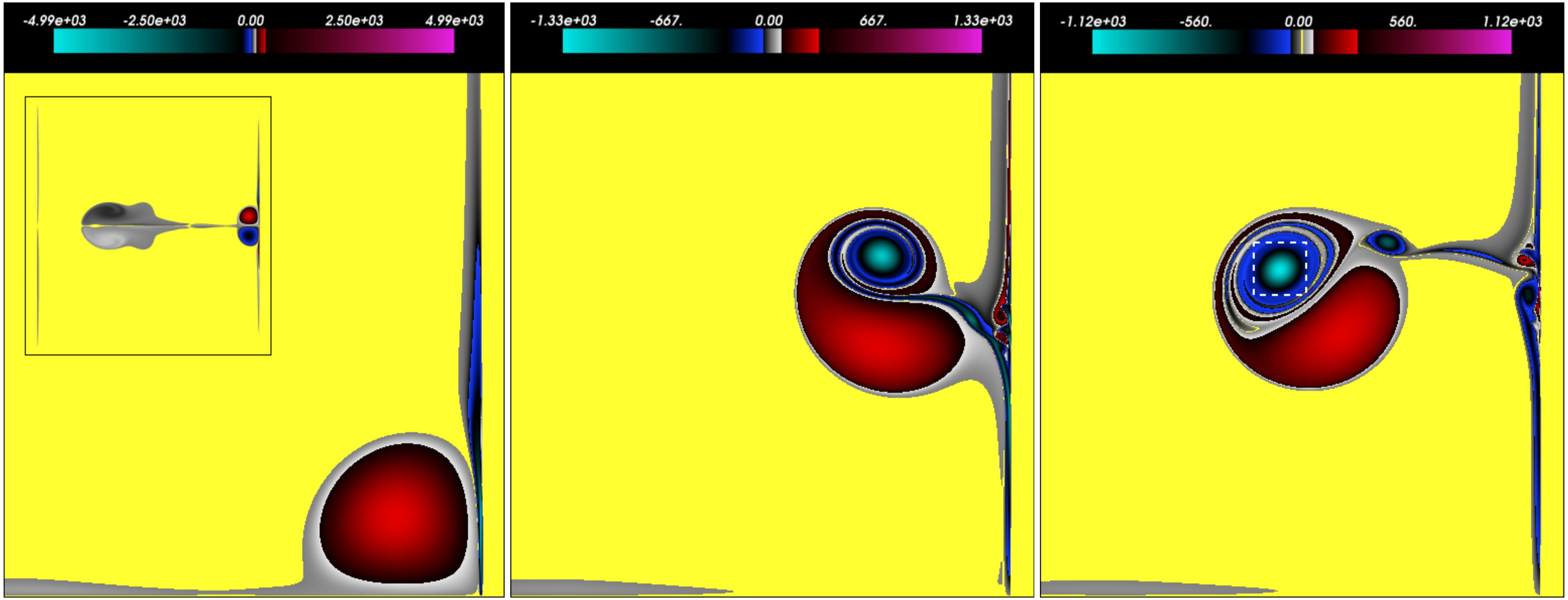
$$\partial_t e + \mathbf{u} \cdot \nabla (e + p) = \nu \Delta e \boxed{- \nu |\nabla \mathbf{u}|^2}$$

Local dissipation rate

DNS of dipole crashing onto a wall

Resolution
N=16384^2

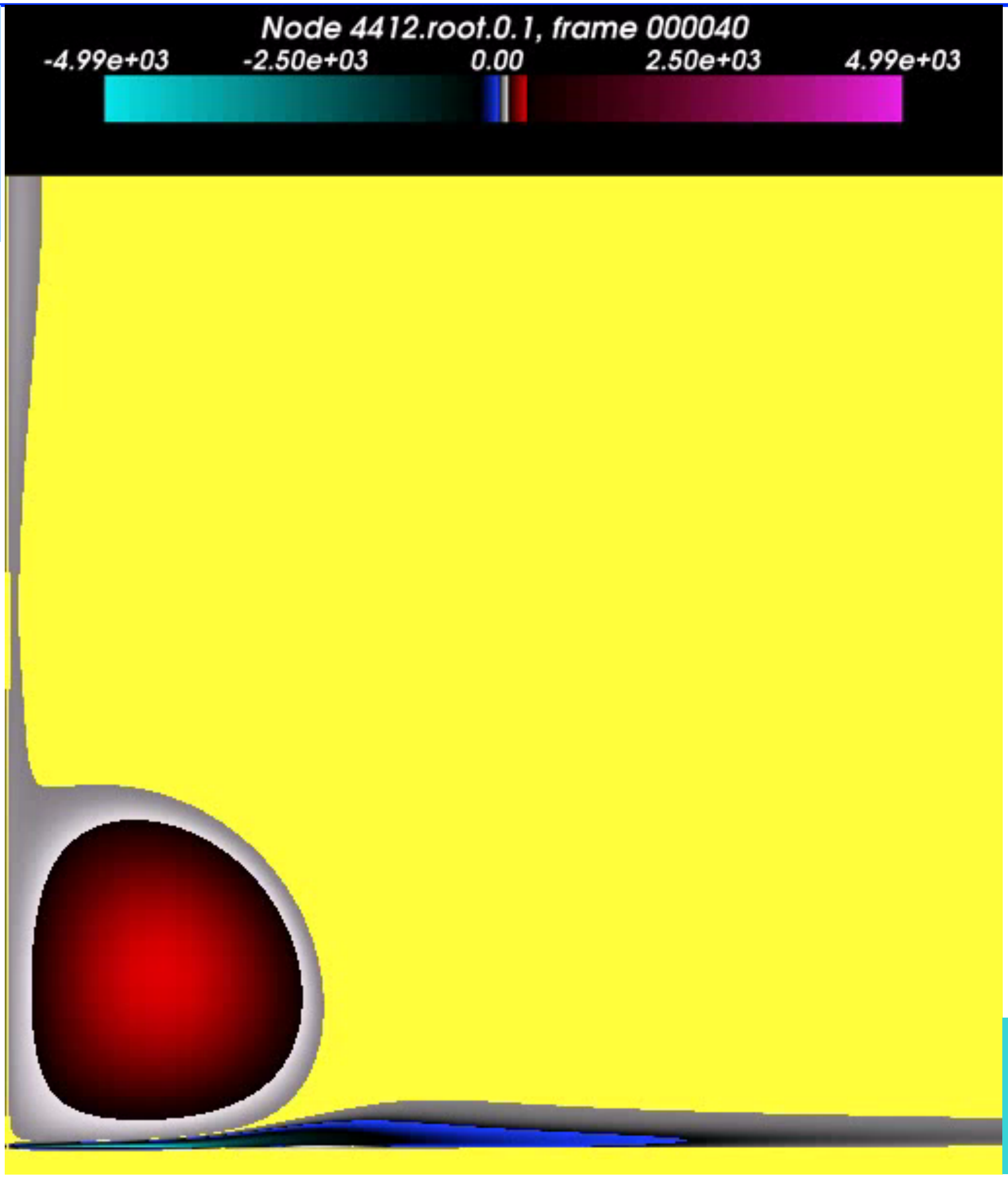
*Nguyen van yen, M. F. and Schneider, PRL, **106**(18)*



t=0.3 *t=0.4* *t=0.5*

Dipole-wall collision at Re=8000

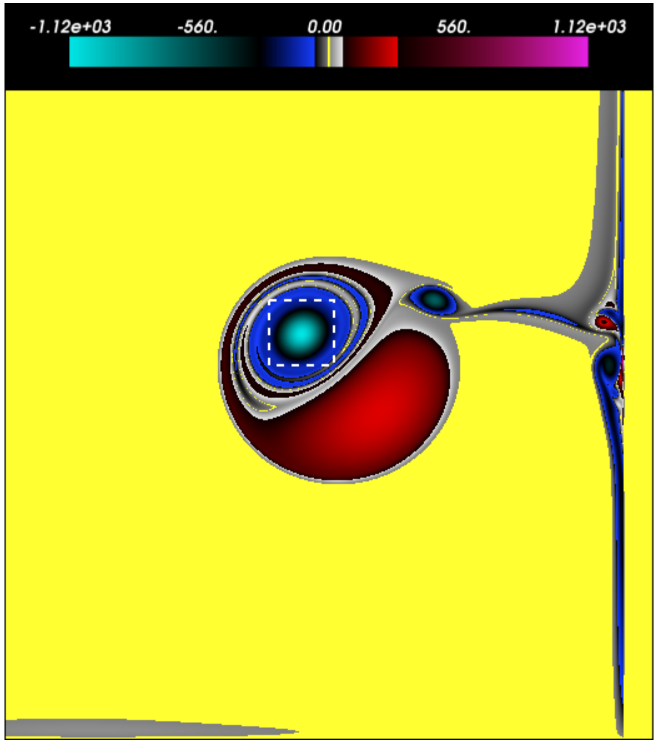
DNS
Resolution
$N=8192^2$

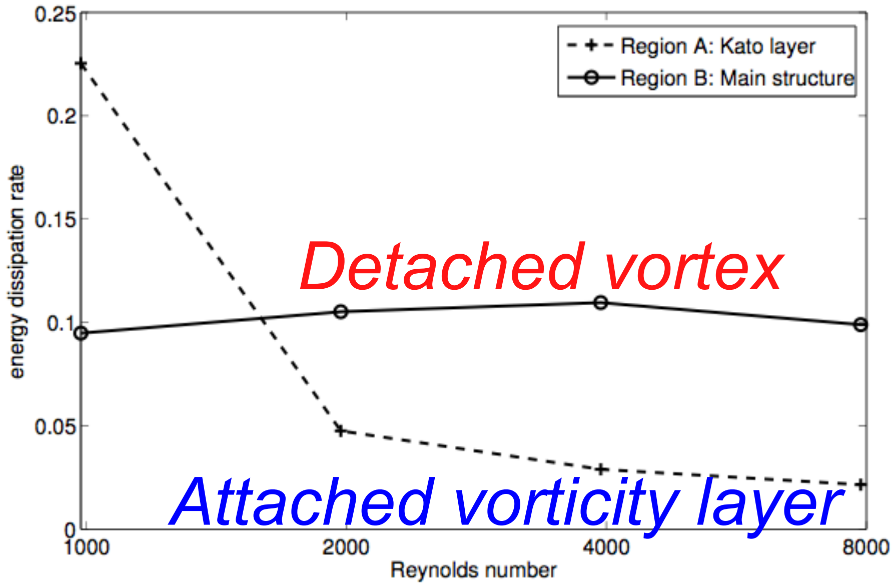


Nguyen van yen, M. F. and Schneider, PRL, ***106****(18)*

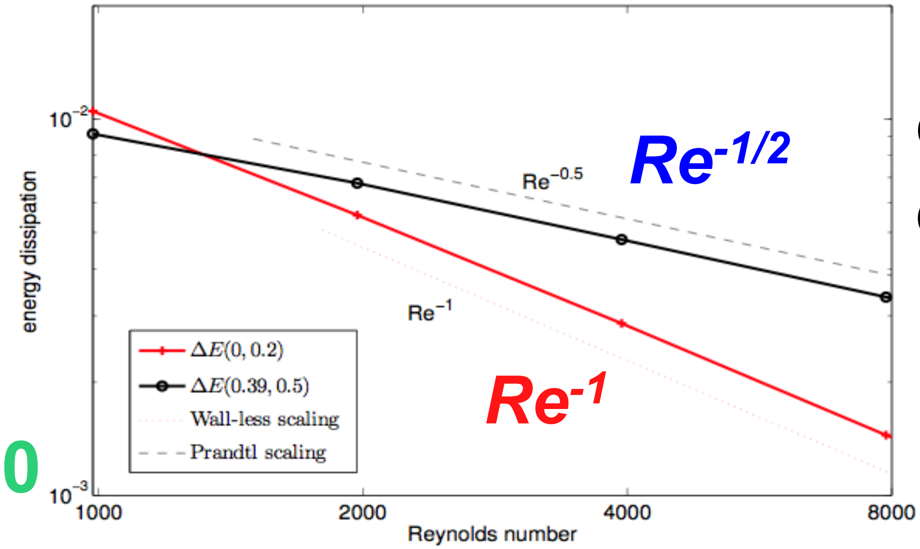
Dissipative structures

*Nguyen van yen, M. F. and Schneider, PRL, **106**(18)*





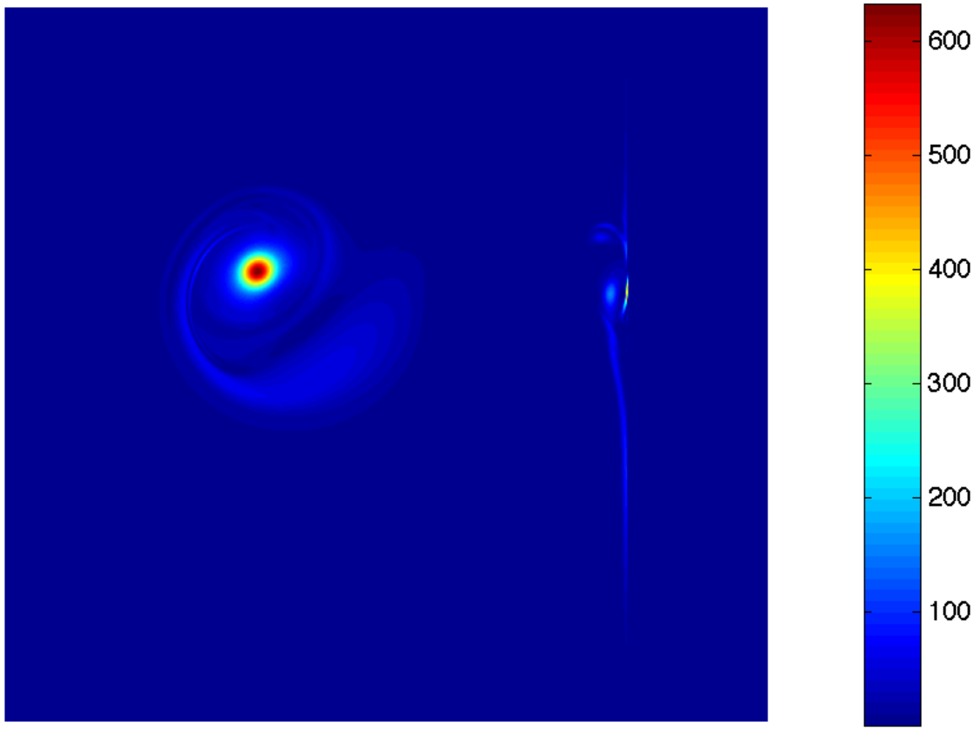
energy dissipation rate



energy dissipation

Re=1000 **Re=8000**

Snapshot of the local dissipation rate



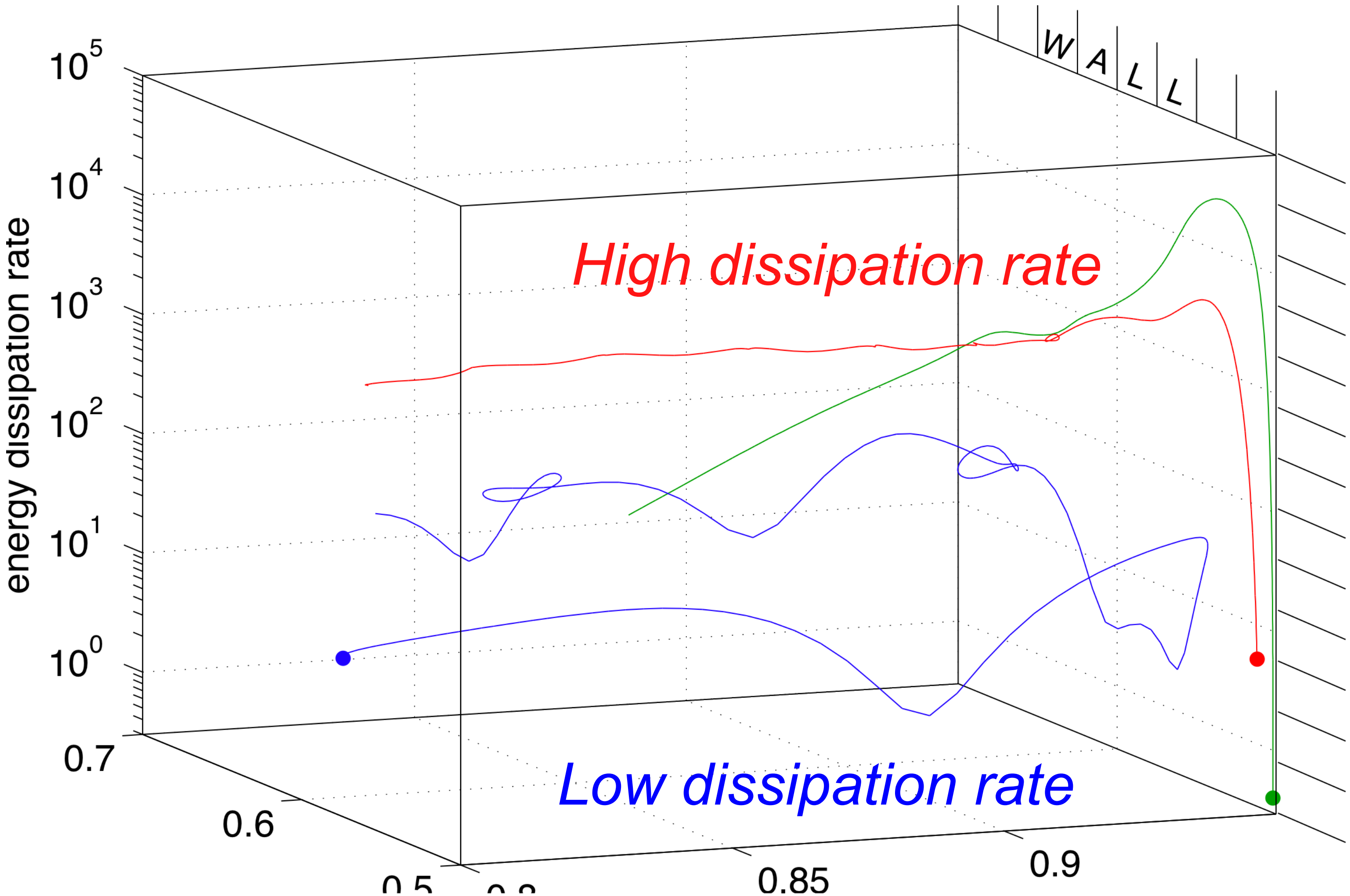
Local dissipation rate
for the dipole-wall collision
at t= 0.5

The strongest values of the energy dissipation rate is observed inside the main vortex that detached from the boundary layer,

rather than inside the boundary layer itself.

R. Nguyen van yen, M. F. and K. Schneider, PRL, ***106****(18)*

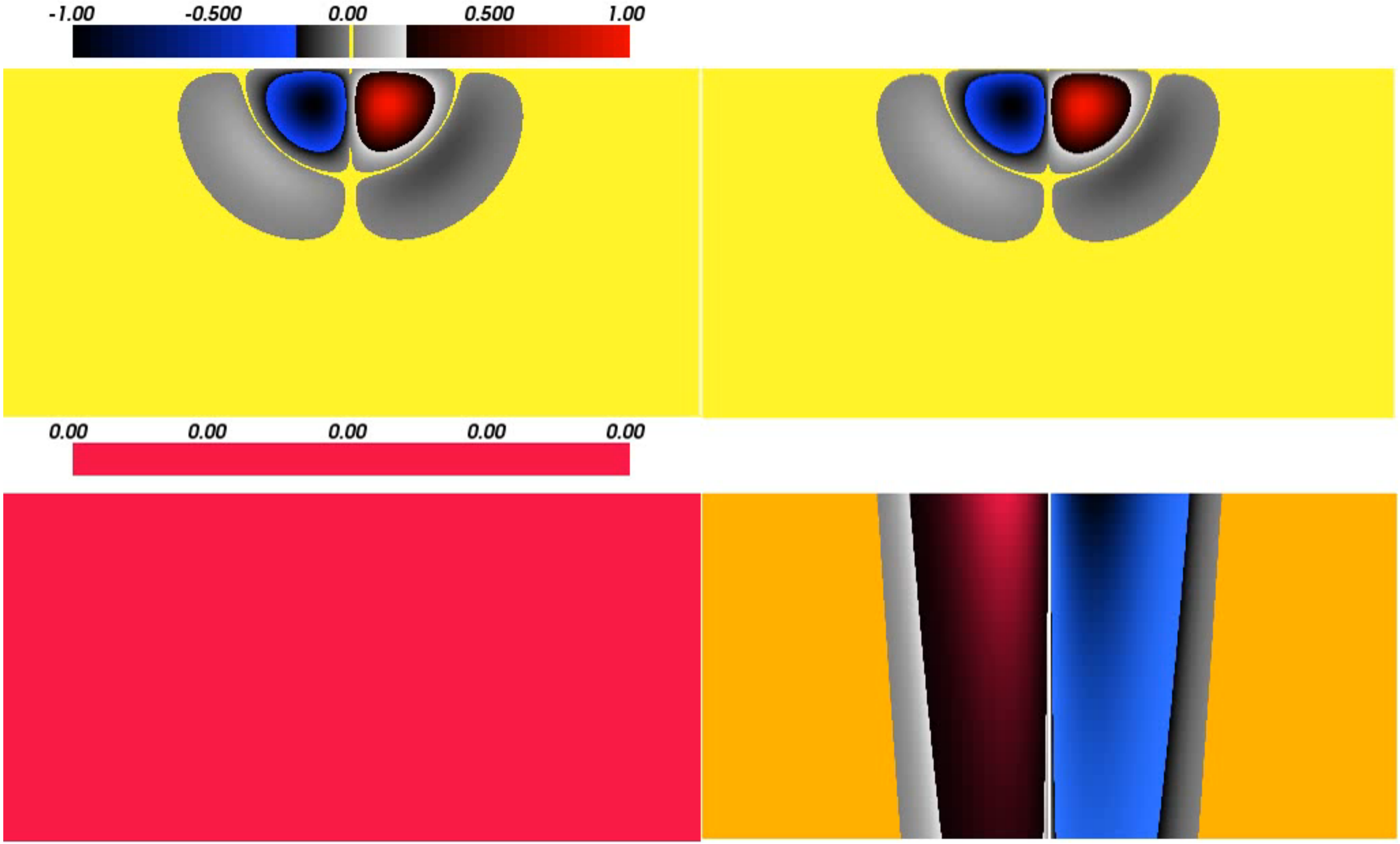
Production of dissipative structures



*R. Nguyen van yen, M. F.
and K. Schneider,
PRL, **106**(18)*

Euler-Prandtl

Navier-Stokes



Prandtl's singularity

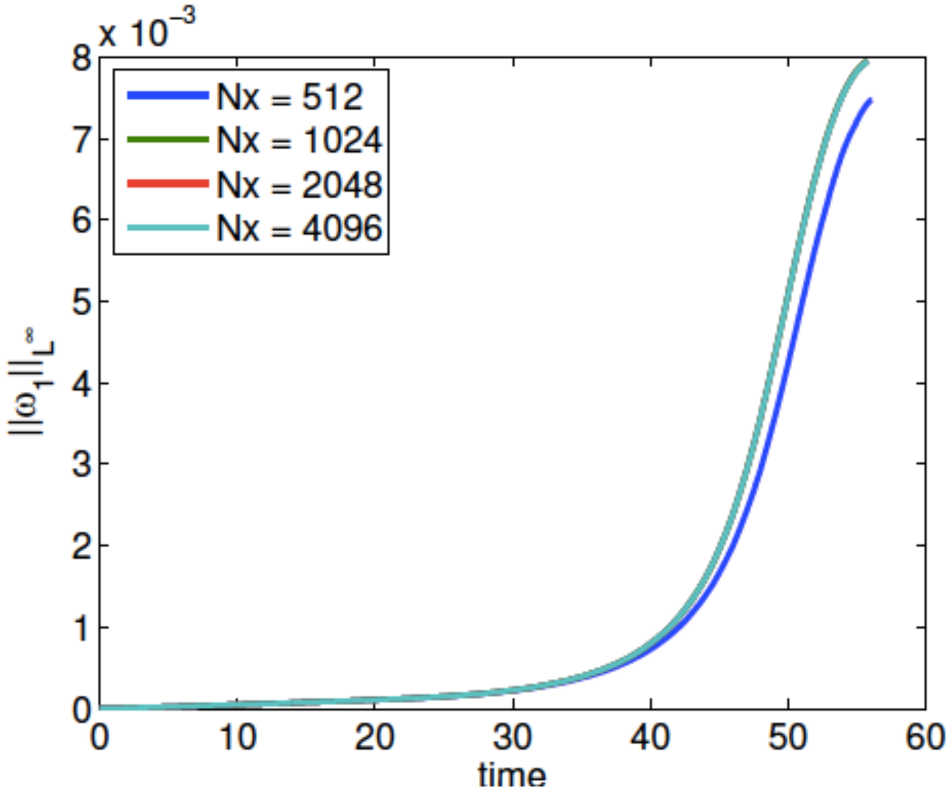
Prandtl equation has well-known finite time singularity

- $|\partial_x \omega_1|$ and $u_{1,y}$ blows up,
- ω_1 remains bounded.

L. L. van Dommelen and S. F. Shen., 1980
J. Comp. Phys., ***38***(2)

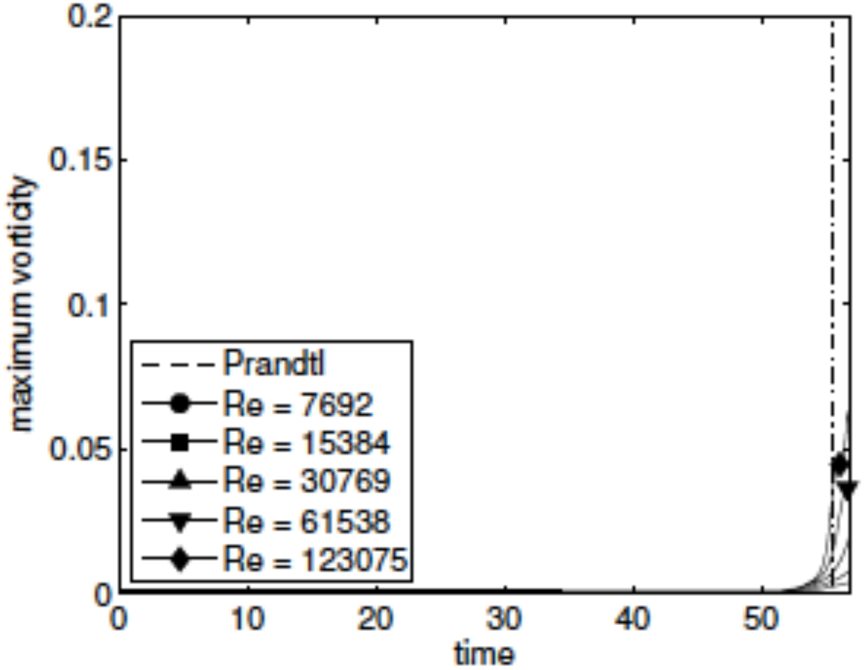
This is observed in our computations as expected,

for $t \to t_D \simeq 55.8$

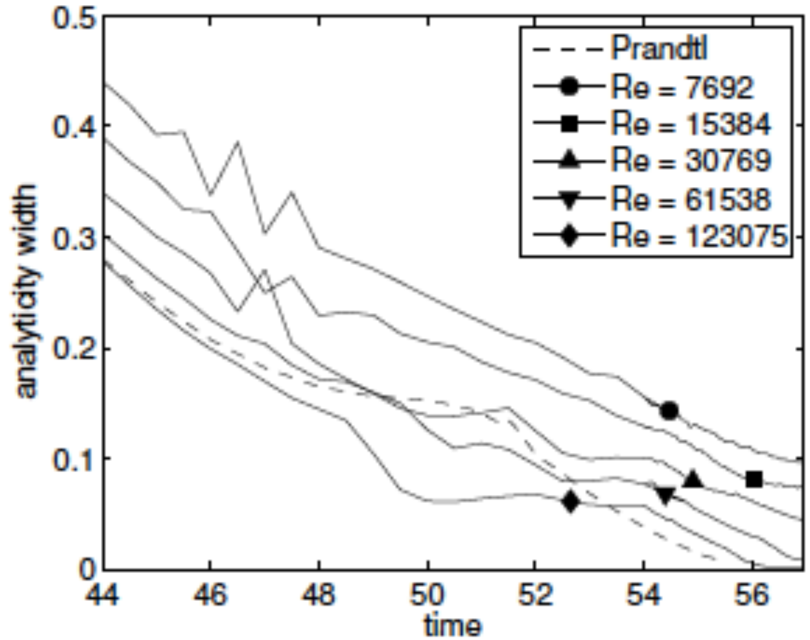


Prandtl solution's blow-up

According to Kato's theorem, and since ω_1 remains bounded uniformly until t_D, we expect that $\mathbf{u}_\nu \xrightarrow[\nu\to 0]{L^2} \mathbf{u}_0$ uniformly on $[0, t_D]$.



Evolution of vorticity max

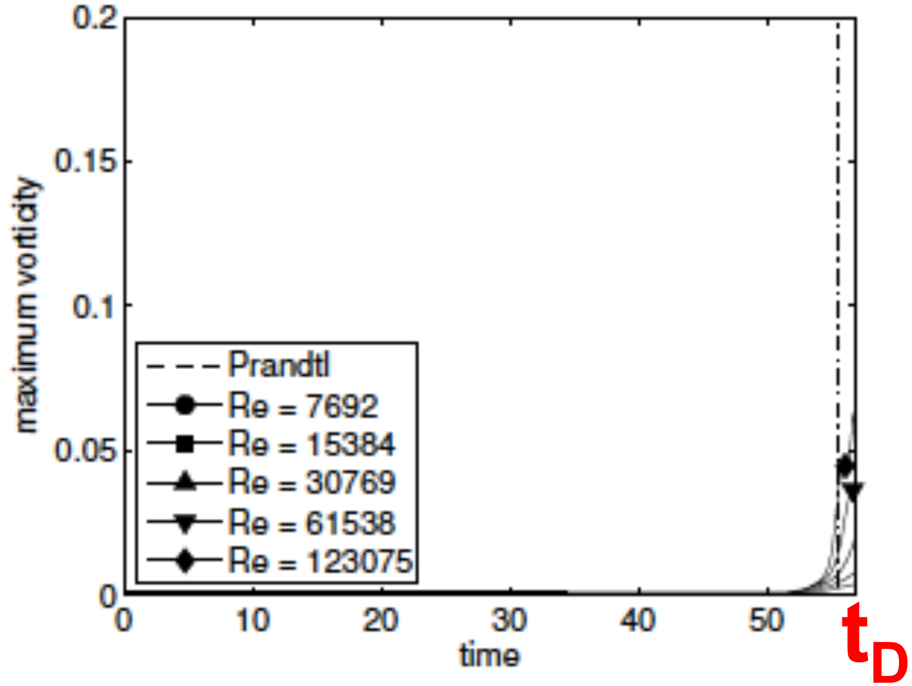


Evolution of analyticity strip

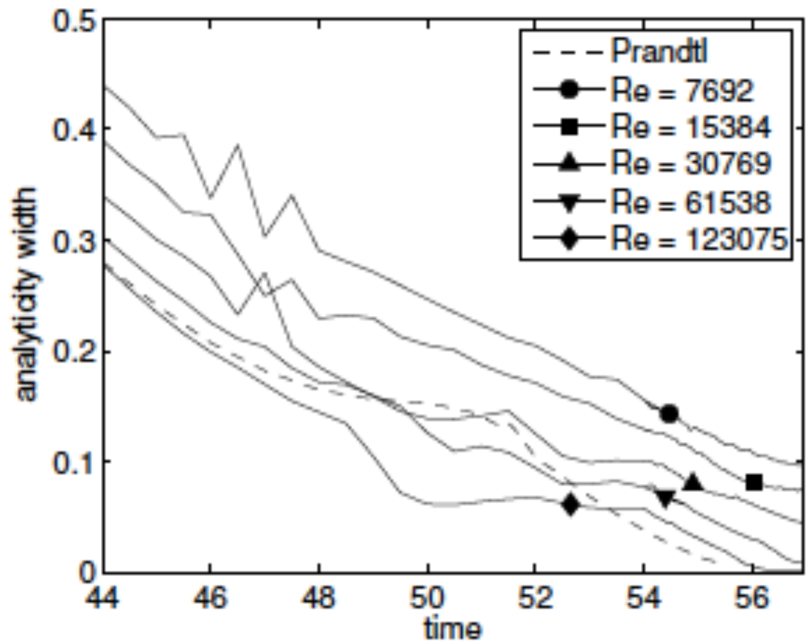
Show convergence!

Prandtl solution blows up at t_D

According to Kato's theorem, and since ω_1 remains bounded uniformly until t_D, we expect that $\mathbf{u}_\nu \xrightarrow[\nu \to 0]{L^2} \mathbf{u}_0$ uniformly on $[0, t_D]$.



Evolution of vorticity max

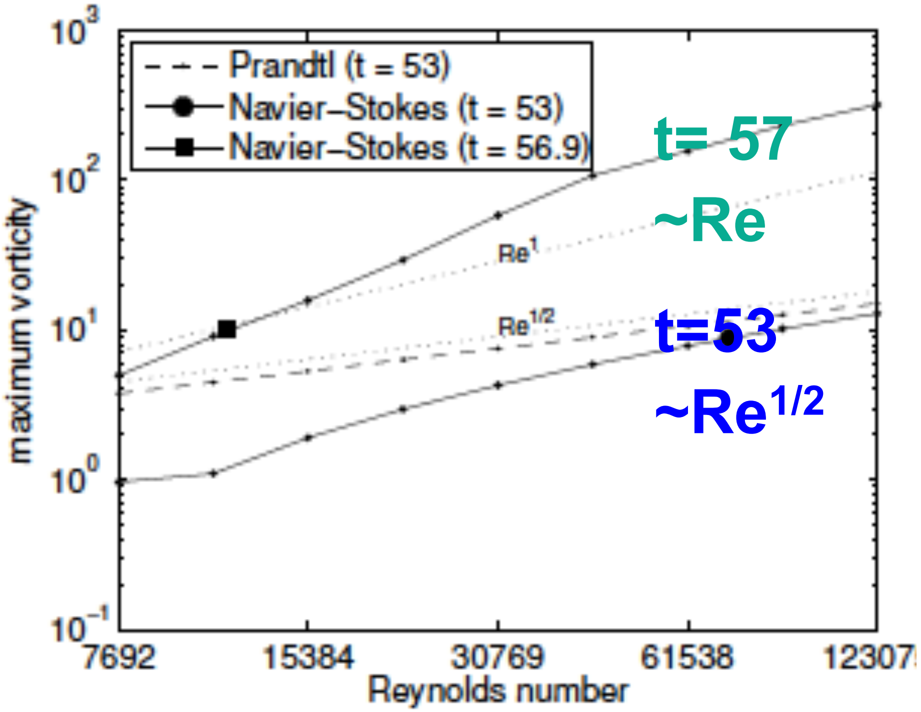


Evolution of analyticity strip

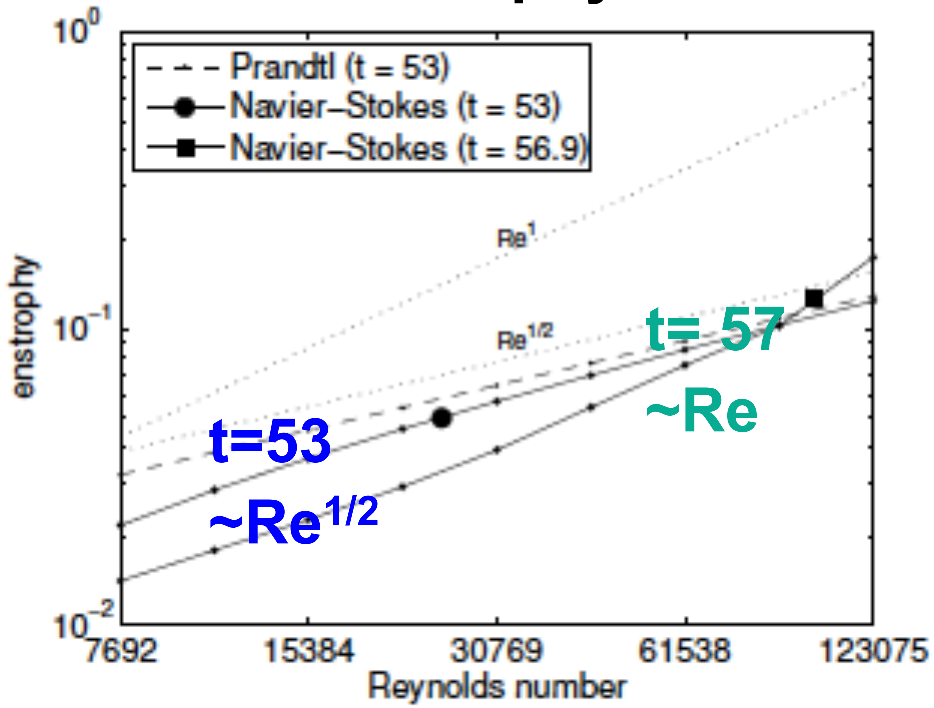
Navier-Stokes solution converges towards Euler's solution for $\nu \rightarrow 0$ until t_D when the bounday layer detaches

What happens after the singularity?

Maximum of vorticity



Enstrophy



Prandtl's scaling in $Re^{1/2}$ before $t_D \sim 55.8$

and Kato's scaling in Re after

Conclusion

The production of dissipative structures is a key feature of fully-developed turbulent flows due to boundary layer detachment.

Prandtl solution becomes singular when boundary layers detach.

The viscous Navier-Stokes solution converges uniformly to the inviscid Euler solution for $t<t_D$, following Prandtl's scaling as $Re^{-1/2}$ but ceases to converge for $t>t_D$.

The detachment process involves spatial scales in different directions, and not only parallel to the wall, as fine as Re^{-1} following Kato's scaling.

*R. Nguyen van yen, M. F. and K. Schneider, 2011 Phys. Rev. Lett., **106**(18), 184502*

R. Nguyen van yen, M. Waidman, R. Klein, M.F. and K. Schneider, 2014 Preprint

Open questions

Numerical results suggest that a new asymptotic description of the flow beyond the breakdown of the Prandtl regime is possible, and studying it might help to understand the observed scalings.

Here are few open questions related to this:

- Would Navier-Stokes solution loose smoothness after t_D and converge to a weak singular dissipative solution of Euler's equation, as suggested by Leray in 1934?
- How can such a weak solution be approximated numerically?

J. Leray, 1934
Sur le mouvement d'un fluide visqueux,
Acta Mathematica, ***63***

C. de Lellis and L. Székzlyhidi, 2010
Archives Rational Mechanics and Analysis,
195*(1), 221-260*

Open mathematical question since 1847

On 16 May 1748 Euler, president of the Prussian Academy of Sciences, read the problem he proposed for the Prize of Mathematics to be given in 1750 :

'Theoria resistentiae quam patitur corpus in fluido motum, ex principiis omnino novis et simplissimis deducta, habita ratione tum velocitatis, figurae, et massae corporis moti, tum densitatis & compressionis partium fluidi'.

Six mathematicians, including d'Alembert, sent a manuscript, but Euler was not satisfied and postponed the prize.

Grimberg, *D'Alembert et les équations aux dérivées partielles en hydrodynamique, Thèse de Doctorat, Université de Paris VII, 1998*

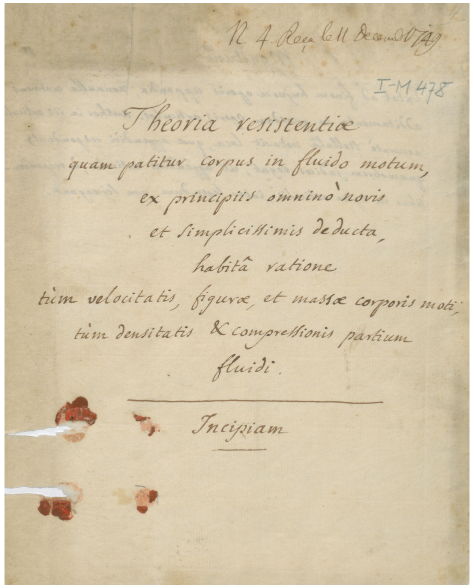
Jean Le Rond d'Alembert (1717-1783)

Leonhard Euler (1707-1783)

D’Alembert’s paradox

D’Alembert was upset and decided to translate his latin Manuscript of 1749 and publish it in French in 1752

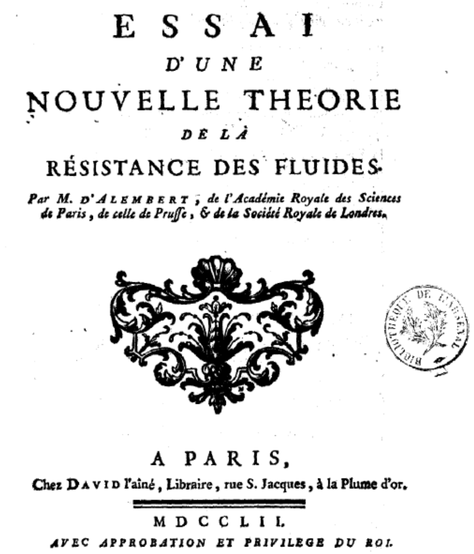
1749



I-M 478

Theoria resistentiæ
quam patitur corpus in fluido motum,
ex principiis omninò novis
et simplicissimis deducta,
habitâ ratione
tum velocitatis, figuræ, et massæ corporis moti,
tum densitatis & compressionis partium
fluidi.

Incipiam



ESSAI
D'UNE
NOUVELLE THEORIE
DE LA
RÉSISTANCE DES FLUIDES.
Par M. D'ALEMBERT, de l'Académie Royale des Sciences de Paris, de celle de Prusse, & de la Société Royale de Londres.

A PARIS,
Chez DAVID l'aîné, Libraire, rue S. Jacques, à la Plume d'or.
MDCCLII.
AVEC APPROBATION ET PRIVILEGE DU ROI.

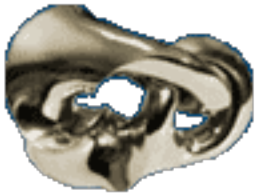
1752

’It seems to me that the theory, developed in all possible rigor, gives, at least in several cases, a strictly vanishing resistance, a singular paradox which I leave to future geometers to elucidate.’

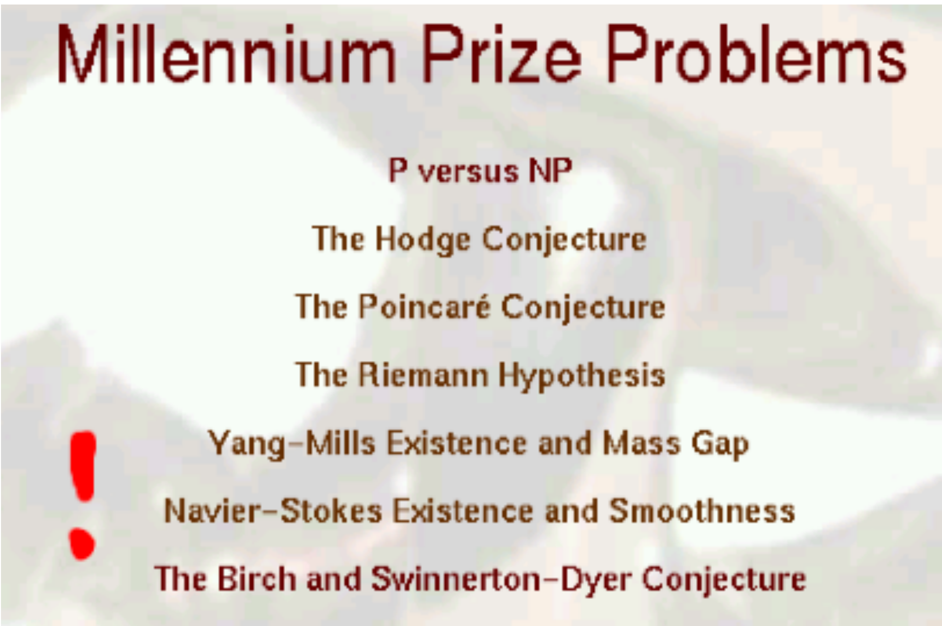
How do Navier-Stokes solutions behave?

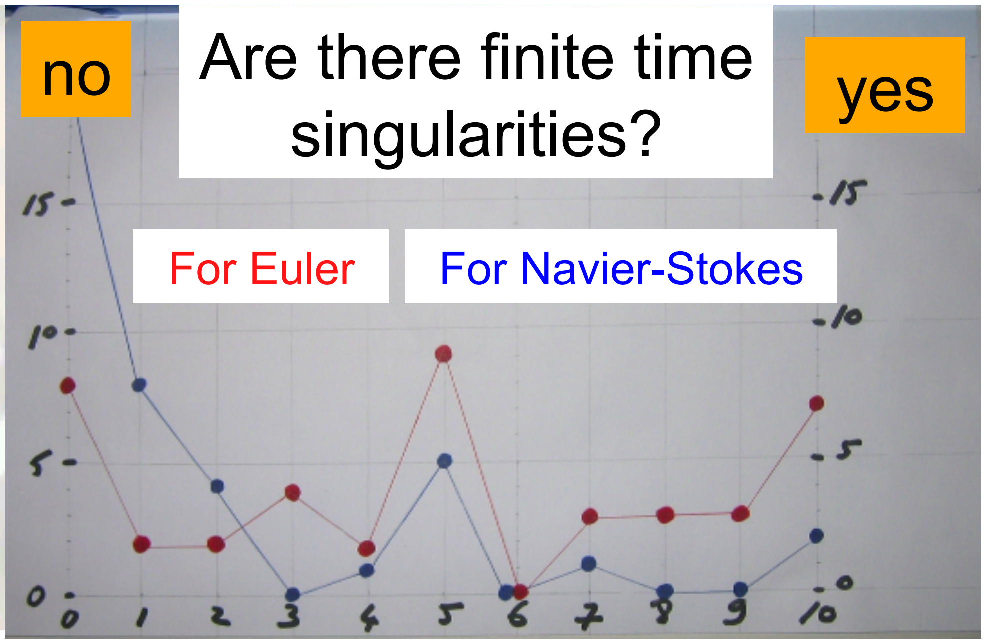
This is still an open problem

$\Longrightarrow$ Clay Prize of Mathematics, 2000 :



'The challenge is to make substantial progress toward a mathematical theory which will unlock the secrets hidden in the Navier-Stokes equations.'





//www.claymath.org/millennium/index.php

M. Otelbayev, 2013

T. Tao, 2014

http://wavelets.ens.fr

You can download movies from :
'Results'

You can download papers from :
'Publications'

You can download codes from:
'Codes'