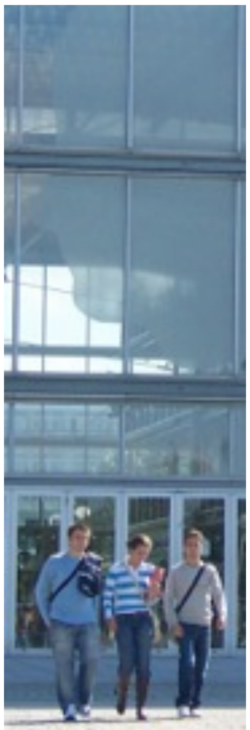
Quasi-geostrophic approximation and a fractional vorticity equation

École des Ponts ParisTech



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Ocean Scale Interactions,

a Tribute to Bach-Lien Hua

IFREMER, Plouzané,

23-25/06/2014

Starry Night, Van Gogh, 1889

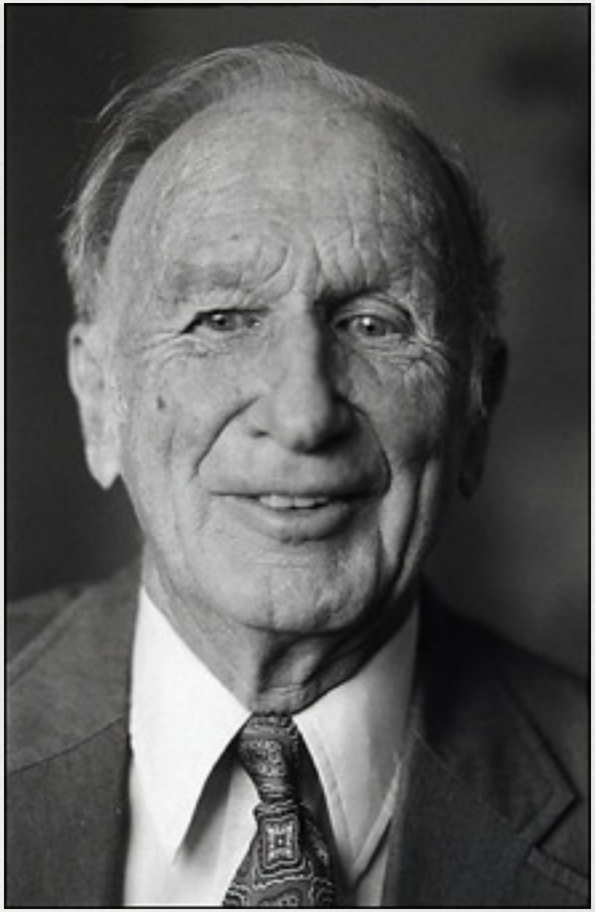
UNIVERSITÉ PARIS-EST

Distinguished Lorenz Lecture

AGU, San Francisco,

5 December, 2012

The Lorenz Lecture



- *The Lorenz Lecture provides an introduction to the role of nonlinearity, complexity, and scaling in earth science processes.*
- *The lecture is given at the Fall meeting (since 2001).*
- *It is named in honor of Edward N. Lorenz, the founder of modern chaos theory and an early contributor to theoretical climate science.*

The 2006 Lorenz Lecture

- *Nonlinear Dynamics of Zonal Jets in Planetary Atmospheres and Oceans*
- ***Bach Lien Hua***
- *LOP, IFREMER, Brest, France*

The 2006 Lorenz Lecture

- *Nonlinear Dynamics of Zonal Jets in Planetary Atmospheres and Oceans*
- ***Bach Lien Hua***
- *LOP, IFREMER, Brest, France*
- ***Lien suddenly passed away on the eve of AGU 2012***

Bach Lien Hua

- *Lien graduated from Ecole Normale Supérieure, State Doctorat at Muséum National d'Histoire Naturelle*
- *worked at Woodshole, NCAR and Hawai, at IFREMER since 1985, vice-director of LPO*
- *outstanding contributions to the understanding of oceanic dynamics: theory and large scale simulation*
- *looking forward, her collaborators suggest to consider a topical conference on these theme, especially for young researchers*

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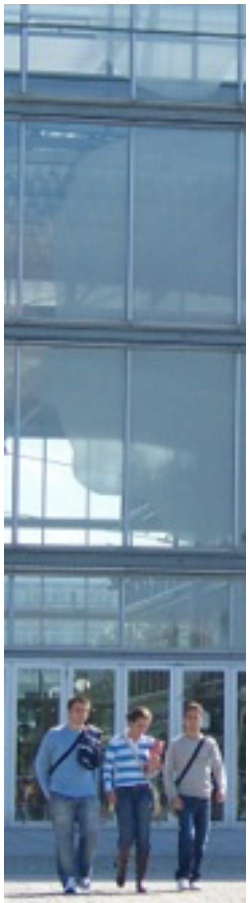
Scale and scaling analyses



At large scales, the total vorticity $\underline{\zeta}$ is dominated by Earth's angular velocity $\underline{\Omega}$: $\underline{\zeta} \approx 2\,\underline{\Omega}$.

$$D(\underline{\zeta})/Dt = ((\underline{\zeta}).\nabla)u$$

(barotropic vorticity equation)

Scale analysis (Charney, 1948; Pedlosly, 1971):

- keep *relevant terms at a given scale*,
- then analyze the scaling behavior (*) of the resulting approximation (Charney, 1971)
- it may be quite different from that of the original equation, e.g. a scaling break.

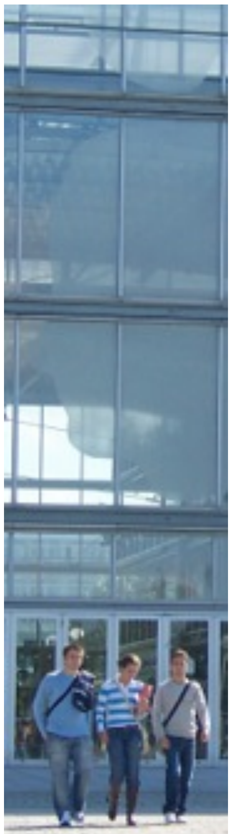
Scaling analysis:

- use scaling to keep as many possible nonlinear interacting terms,
- anisotropy may introduce a *statistical symmetry breaking* of the original equations *across scales*
- and yields a new set of equations with *non trivial symmetries*.

Université Paris-Est

(*) not so straightforward, Herring (1980), Hua et Haidvogel (1986)

Van der Hoven wind (integrated) spectrum (1957)



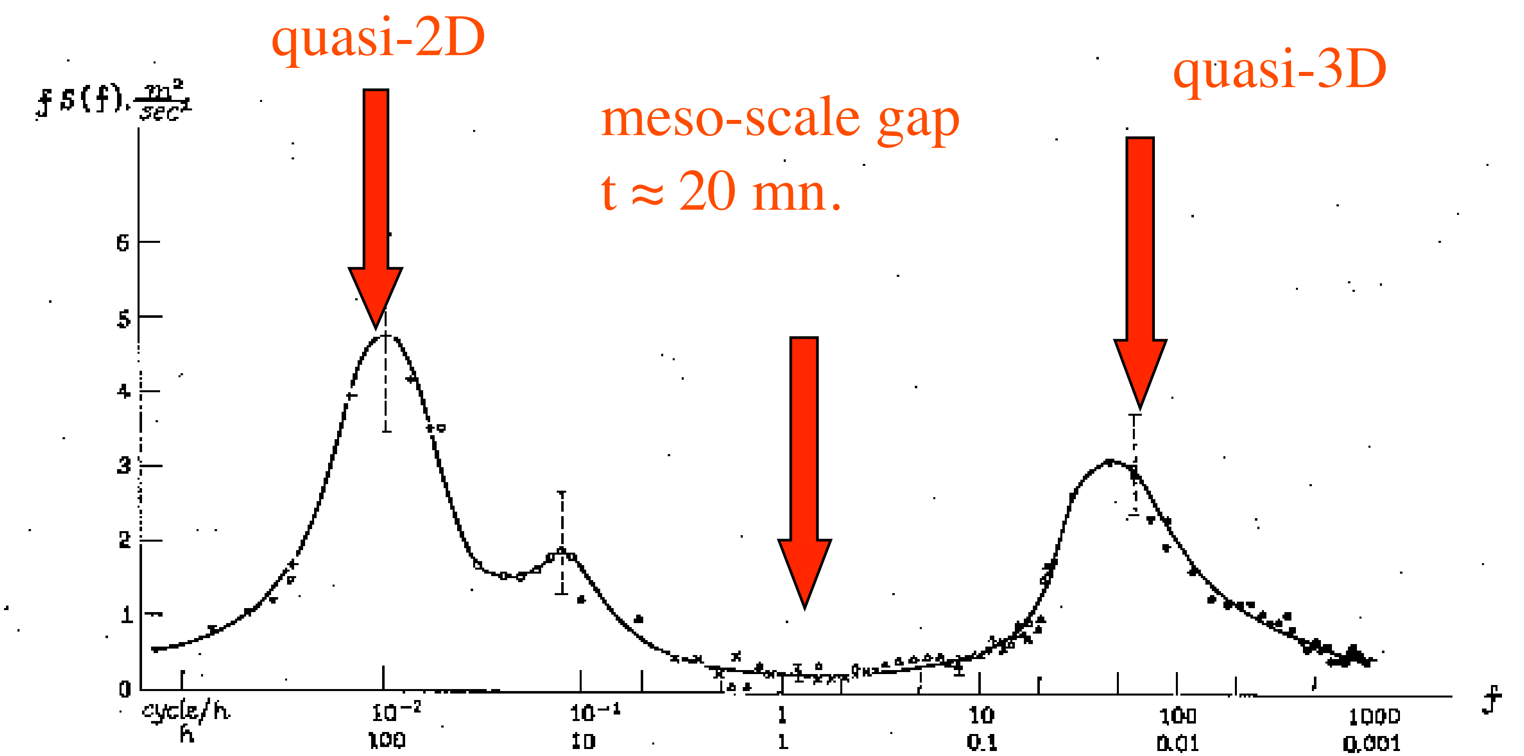
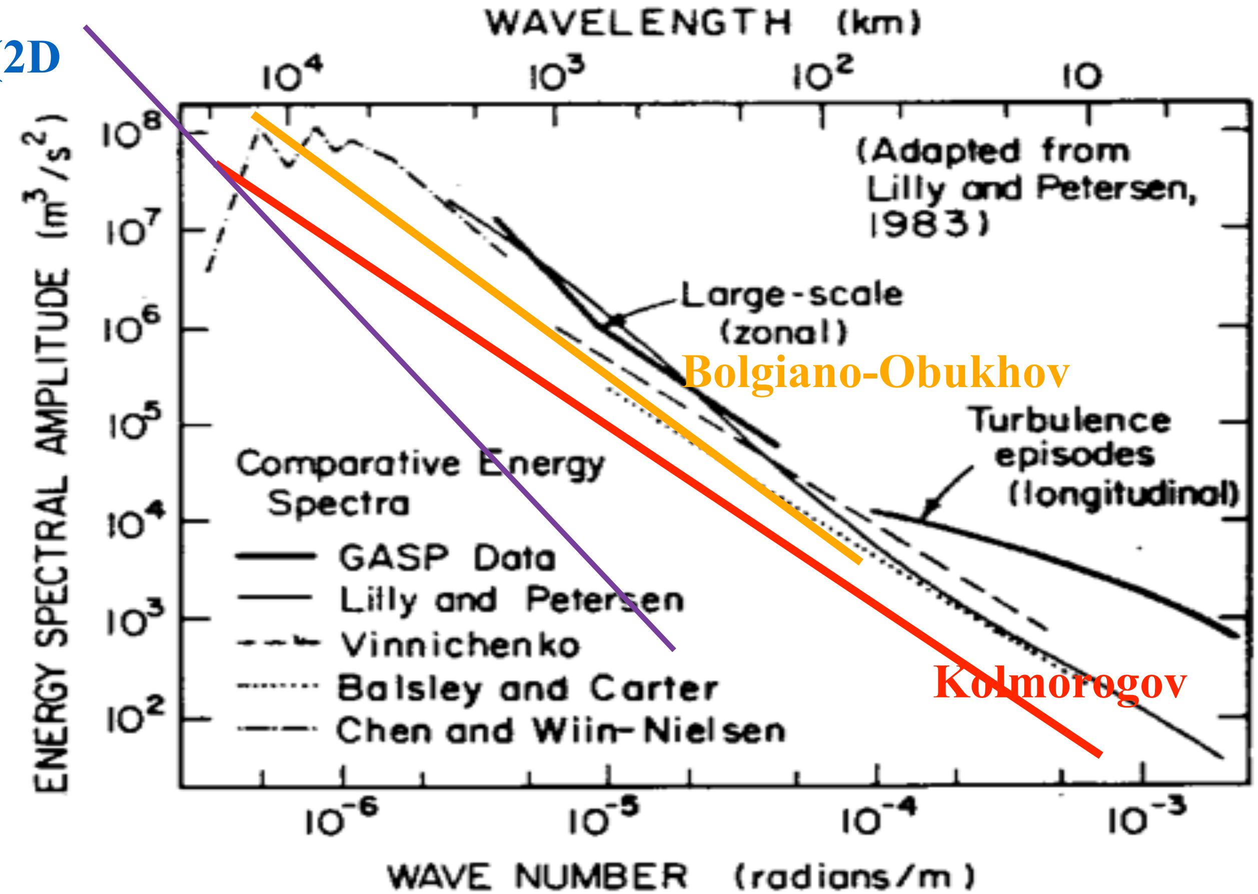


Fig. 3 Spectrum of the horizontal wind velocity. After Van der Hoven.[26] Some experimental points are shown on the graph; see reference 26.

Richardson cascade is split into macro, meso, micro oscillations...

GASP Experiment

Scaling of the horizontal shear of the horizontal wind



Adapted from Nastrom and Gage 1983

Unique measurement device:

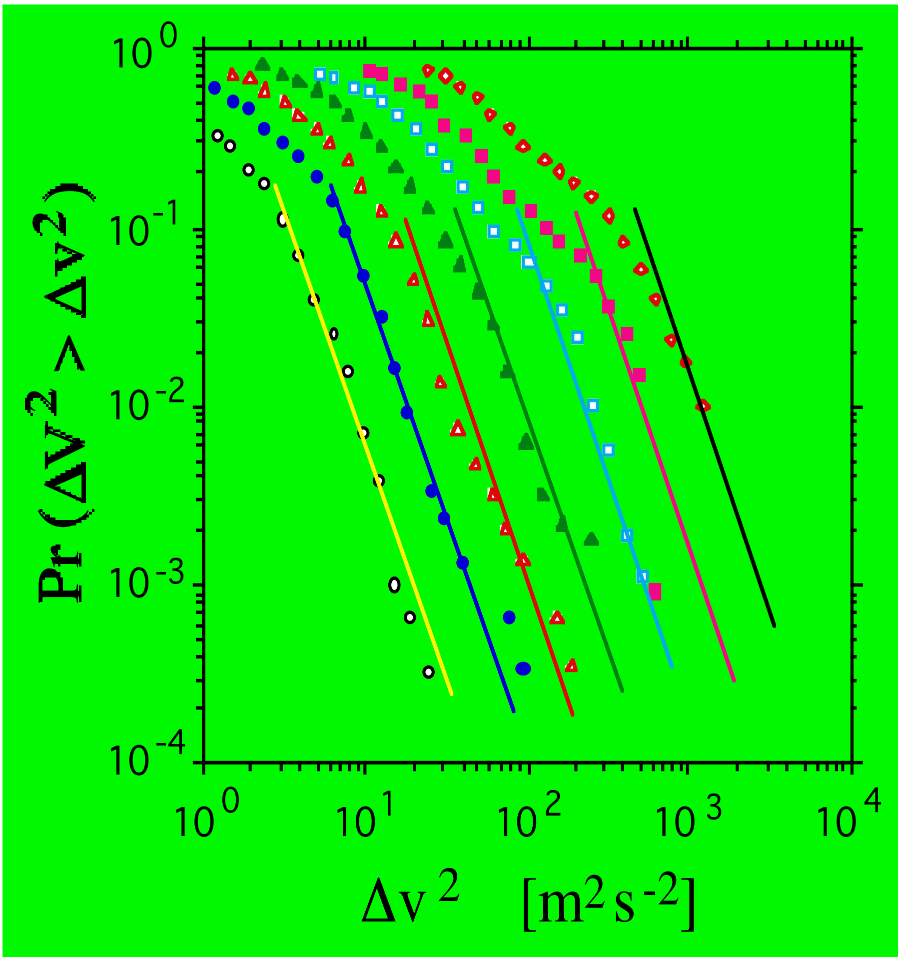
- airplane deviations

- **no longer any evidence of a spectral gap !**
- **rather K41 scaling**

$$\Delta v_h(\Delta x) \approx \Delta x^{1/3}$$

Landes experiment (Météo-France)

Scaling of the vertical shear of the horizontal wind



S+ L,TSF4, 1984

80 balloon vertical soundings: the probability distribution of the vertical shears of the horizontal wind is scaling for Δ = 50,100,200, 400,800, 1600, 3200 m (*):

$$\Delta v_h(\Delta z) \approx \Delta z^{3/5}$$

(Bolgiano-Obukhov scaling)

power law tails: $q_D \approx 5$!

(*) see also Endlich et al., JAS, 1969, Adelfang, JAS, 1971, Lazarev+al, NPG, 1994, Hovde, JGR, 2002, Tuck, 2003, Lilley +al, PRE, 2004, Radkevich, Physica A, 2007, L+al, 2009..

Pullback/pushforward transforms

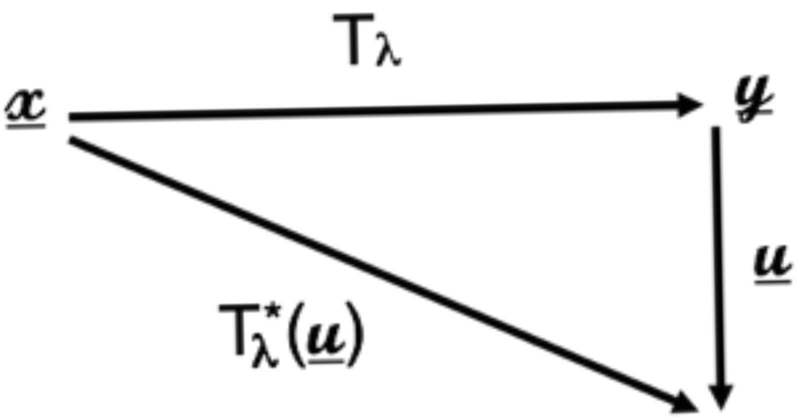
Contraction ($\lambda>1$) /dilation ($\lambda>1$) of time and/or space (1-parameter group):

$$T_\lambda\, x = x/\lambda\,; \quad T_{\lambda\lambda'} = T_\lambda\, T_{\lambda'}$$

- **pullback function**(contravariant)
  - $T_\lambda^*: f \;\text{-->}\; T_\lambda^*\, f(.) = f\,(T_\lambda .)$
- **pushforward measure** (covariant)
  - $T_{\lambda,*} : \mu \;\text{-->}\; T_{\lambda,*}\, \mu$

$$\int T_\lambda^*(f)\,d\mu = \int f\, dT_{\lambda,*}(\mu)$$

(in particular : $f=1_B$)



$$T_\lambda^*(\underline{u})(\underline{x}) = \underline{u}(\underline{y}) = \underline{u}(T_\lambda(\underline{x}))$$

=> T_λ^* and $T_{\lambda,*}$ 1-parameter groups (with possibly random generators):

$$T_{\lambda,*}\mu = \lambda^\Gamma \mu$$

- e.g.: regular measure: $\Gamma = d$
  - -Dirac measure: $\Gamma = 0$
  - –fractal measure: $\Gamma = d - C = D$

- Aggregation of μ yields γ!
- => empirical evidence of the singular nature of the rainfall measure.

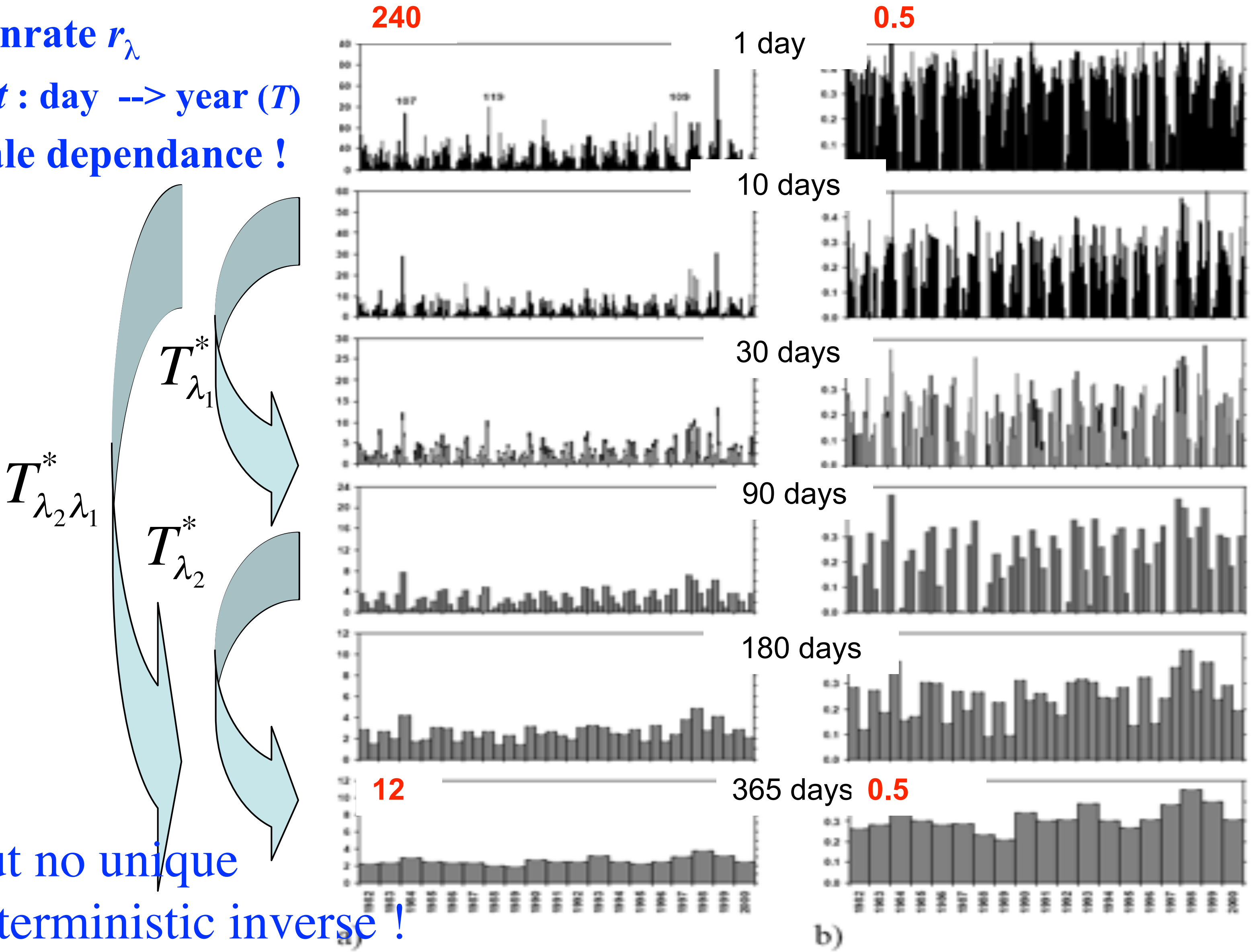
Scale dependency vs. singularities

Resolution: $\lambda = T/\Delta t$

rainrate r_λ

Δt : day --> year (T)

scale dependance !



Singularities γ:

$$\gamma = Log_\lambda(r_\lambda)$$

$$r_\lambda \approx \lambda^\gamma$$

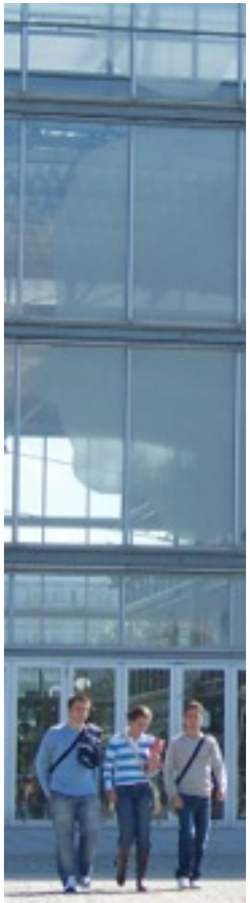
No longer scale dependance !

Singular Measure !

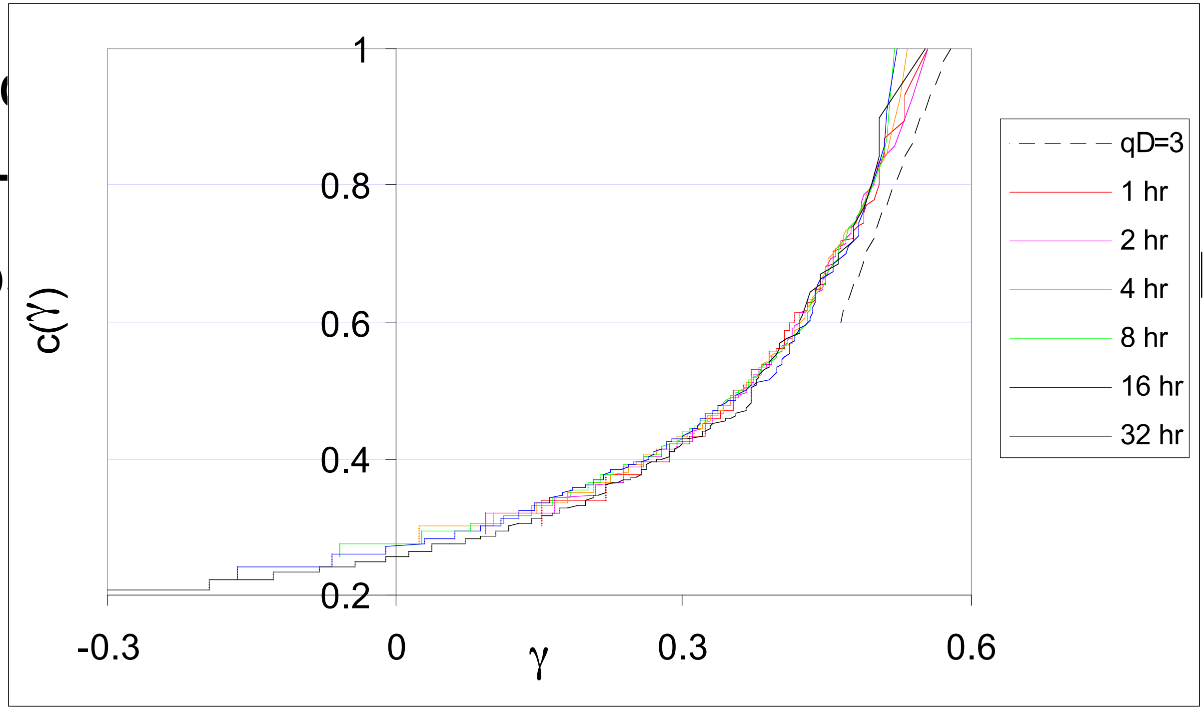
But no unique deterministic inverse !

(S+al, JHS 2010)

Another miracle !

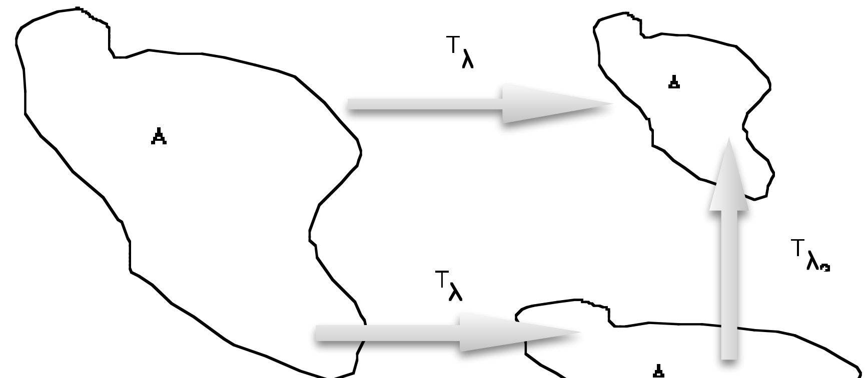


- the (
$c(\gamma)$= -
collap on't!



Generalised Scales and Dilations

First posit scaling,
then study remaining symmetries
≠ local isotropy (Kolmogorov's 41)



$$T_{\lambda'} \circ T_{\lambda} = T_{\lambda'\lambda}$$

Multiplicative (semi-)group of *generalized* contractions/dilations with corresponding *generalized scale*

$$T_{\lambda} = \lambda^{-G} \equiv \exp(-G.Log(\lambda))$$

$$\|\underline{x}\|$$

i) *nondegeneracy*:

$$\|\underline{x}\| = 0 \Leftrightarrow \underline{x} = \underline{0}$$

ii) *linearity with the contraction parameter*

$$\forall \underline{x} \in E, \forall \lambda \in R^{+} :$$

$$T_{\lambda}\|\underline{x}\| \equiv \|T_{\lambda}\ \underline{x}\| = \lambda^{-1}\|\underline{x}\|$$

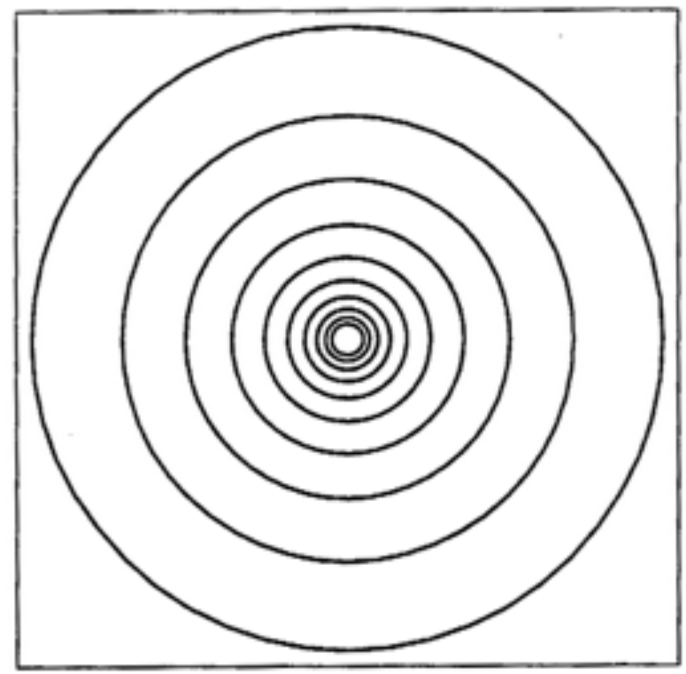
iii) *Balls defined by this scale, must be strictly decreasing with the contraction*

$$B_{\ell} = \{\underline{x} \mid \|\underline{x}\| \leq \ell\} : \ B_{\ell/\lambda} = T_{\lambda}(B_{\ell}) \subset B_{\ell}$$

(S+L, 85; S&al, 2000)

Euclidean scales vs. physical scales
(23/9-D atmospheric turbulence)

Isotropic scales

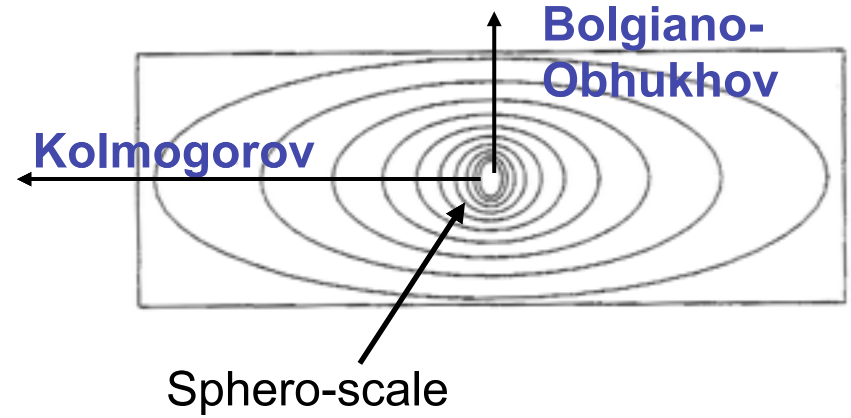


$H_z=1$

- stratification
- flattening of structures at larger scales

$$\|\Delta r\| = l_s\left(\left(\frac{\Delta y}{l_s}\right)^2 + \left(\frac{\Delta x}{l_s}\right)^2 + \left(\frac{\Delta z}{l_s}\right)^{2/H_z}\right)^{1/2}$$

Anisotropic physical scales



$H_z=5/9$

Generalised scale symmetry/invariance

– fundamental importance of symmetries:

plus ça change, plus c'est la même chose! and vice-versa:

dynamics are embodied in symmetries !

$$X \xrightarrow{\varepsilon} Y$$

domain codomain

$$S_\lambda \varepsilon = \varepsilon$$

- scale symmetry/invariance or scaling results from: T_λ
  - pullback a dilation of the domain:
  - rescale the resulting values (contract the codomain): $\tilde{T}_\lambda^{-1}$

$$S_\lambda = \tilde{T}_\lambda^{-1} \circ T_{*\lambda}$$

Vorticity equation

$$D\underline{\omega}/Dt = \underline{s} + \underline{b}; \quad \underline{s} = \underline{\omega} \cdot \nabla \underline{u}; \quad \underline{b} = \nabla \rho \times \nabla(p)/\rho^2$$

stretching (and tilting) vector

baroclinic vector

"3D effects": nonlinear growth
enstrophy catastroph

Rotational symmetry and (scalar) scaling *(for any singularity* γ *)*

$$T^*_\lambda \underline{u} = \lambda^\gamma \underline{u}; \; T^*_\lambda \underline{\omega} = \lambda^{1+\gamma} \underline{\omega}; \; T^*_\lambda \nabla = \lambda \nabla; \; T^*_\lambda D/Dt = \lambda^{1+\gamma} D/Dt$$

Quasi-geostrophic approximation (Charney, 1948)

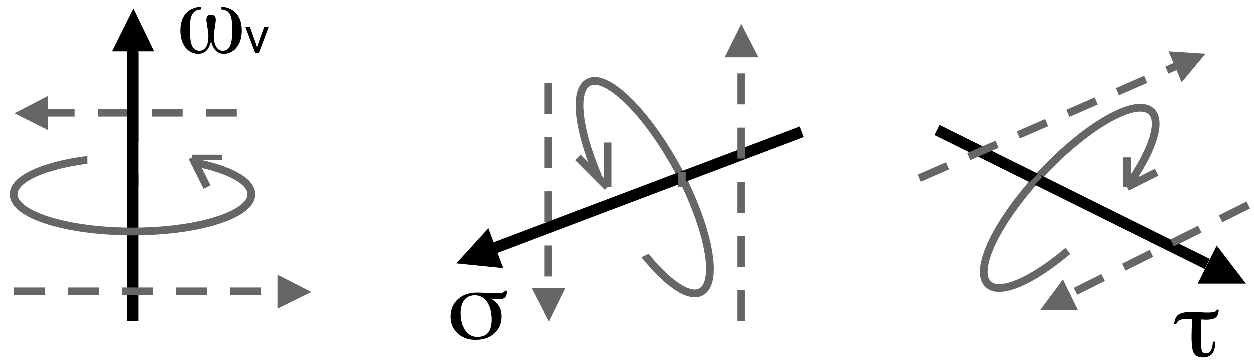
geostrophic material derivative

$$D_g/Dt = \partial/\partial t + \underline{u}_g \cdot \nabla_h = \partial/\partial t + J(\psi, .)$$

linearization of the stretching vector

$$(\underline{\zeta} + f\underline{n}) \cdot \nabla \underline{u} \approx f \partial w/\partial z \approx f_0 \partial w/\partial z$$

Velocity and vorticity decomposition



vertical vorticity
from horizontal shears
of horizontal velocity

horizontal vorticity
from horizontal shears
of vertical velocity

horizontal vorticity
from vertical shear
of the horizontal velocity

Expected anisotropy of vertical vs. horizontal: $\vec{u} = \vec{u}_h + \vec{u}_v; \;\; \vec{\nabla} = \vec{\nabla}_h + \vec{\nabla}_v$

$$\vec{\omega} = \vec{\omega}_v + \vec{\sigma} + \vec{\tau}; \;\; \vec{\omega}_v \equiv \vec{\nabla}_h \times \vec{u}_h; \;\; \vec{\sigma} \equiv \vec{\nabla}_h \times \vec{u}_v; \;\; \vec{\tau} \equiv \vec{\nabla}_v \times \vec{u}_h$$

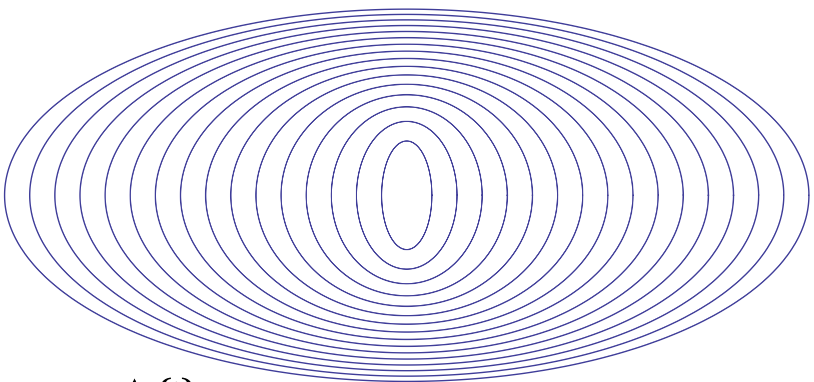
$$\vec{s} = ((\vec{\sigma} + \vec{\tau}) \cdot \vec{\nabla}_h + \vec{\omega}_v \cdot \vec{\nabla}_v)(\vec{u}_h + \vec{u}_v)$$

noting that $(\vec{\sigma} \cdot \vec{\nabla}_h)\vec{u}_v = 0$

Pullback T_{λ^*} and a D-dimensional vorticity equation

D=2+Hz (0<Hz<1)

Stratified atmosphere:



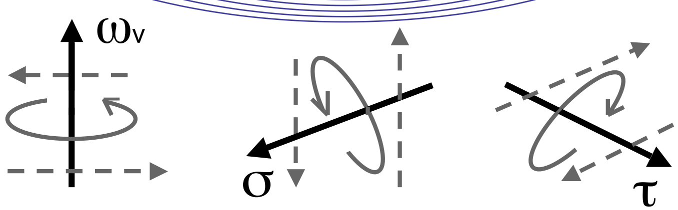
$$T_\lambda = exp(-GLog\lambda) = \lambda^{-G}$$

$$G = diag(g_i);\ g_1 = g_2 = 1;\ g_3 = H_z = 1 - h$$

to preserve incompressibility: $T^*_\lambda \underline{u} = \lambda^\Gamma \underline{u}$

$$\Gamma = diag(\gamma_i);\ \gamma_1 = \gamma_2 = \gamma;\ \gamma_3 = \gamma + h$$

$$\Longrightarrow\ T^*_\lambda(D/Dt) = \lambda^{\gamma+1} D/Dt$$



==> 3 types of scaling for the vorticity

$$T^*_\lambda(\vec{\omega}_v) = \lambda^{1+\gamma}\vec{\omega}_v;\ T^*_\lambda(\vec{\sigma}) = \lambda^{1+\gamma+h}\vec{\sigma};\ T^*_\lambda(\vec{\tau}) = \lambda^{1+\gamma-h}\vec{\tau}$$

the 3D vorticity equations split into 3 equations having distinct scaling behaviors:

$$D\vec{\sigma}/Dt = (\vec{\sigma}\cdot\vec{\nabla}_h)\vec{u}_h$$

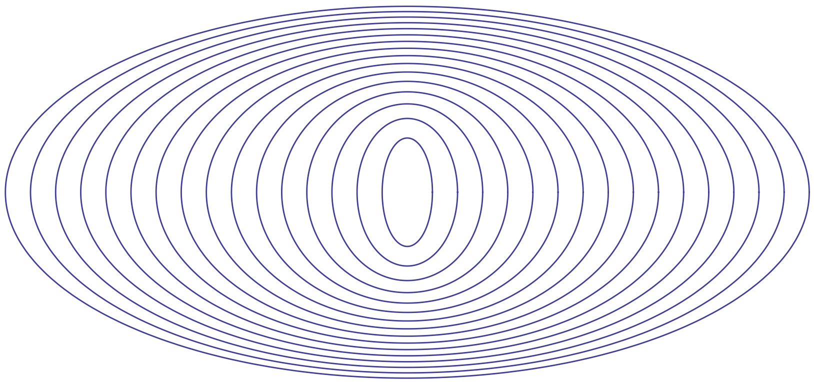
$$D\vec{\tau}/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_h$$

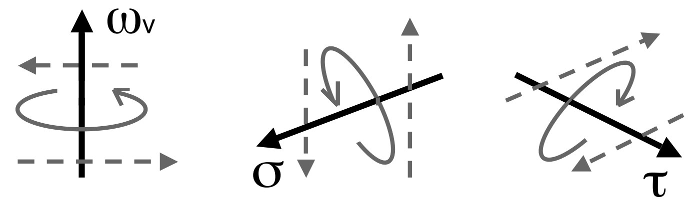
$$D\vec{\omega}_v/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_v$$

D-dimensional vorticity equation

D=2+Hz   (0<Hz<1)

Stratified atmosphere:





$$D\vec{\sigma}/Dt = (\vec{\sigma}\cdot\vec{\nabla}_h)\vec{u}_h$$

$$D\vec{\tau}/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \boxed{\vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_h}$$

$$D\vec{\omega}_v/Dt = (\vec{\tau}\cdot\vec{\nabla}_h + \vec{\omega}_v\cdot\vec{\nabla}_v)\vec{u}_v$$

Strong interactions between *local generalized* scales,

= *strongly non local* (Euclidean) scales !

- a difficulty for direct numerical simulations ?
- easy for stochastic simulations

Fractionnaly Integrated Flux model (FIF, vector version)

FIF assumes that both the renomalized propagator G_R and force f_R are known:

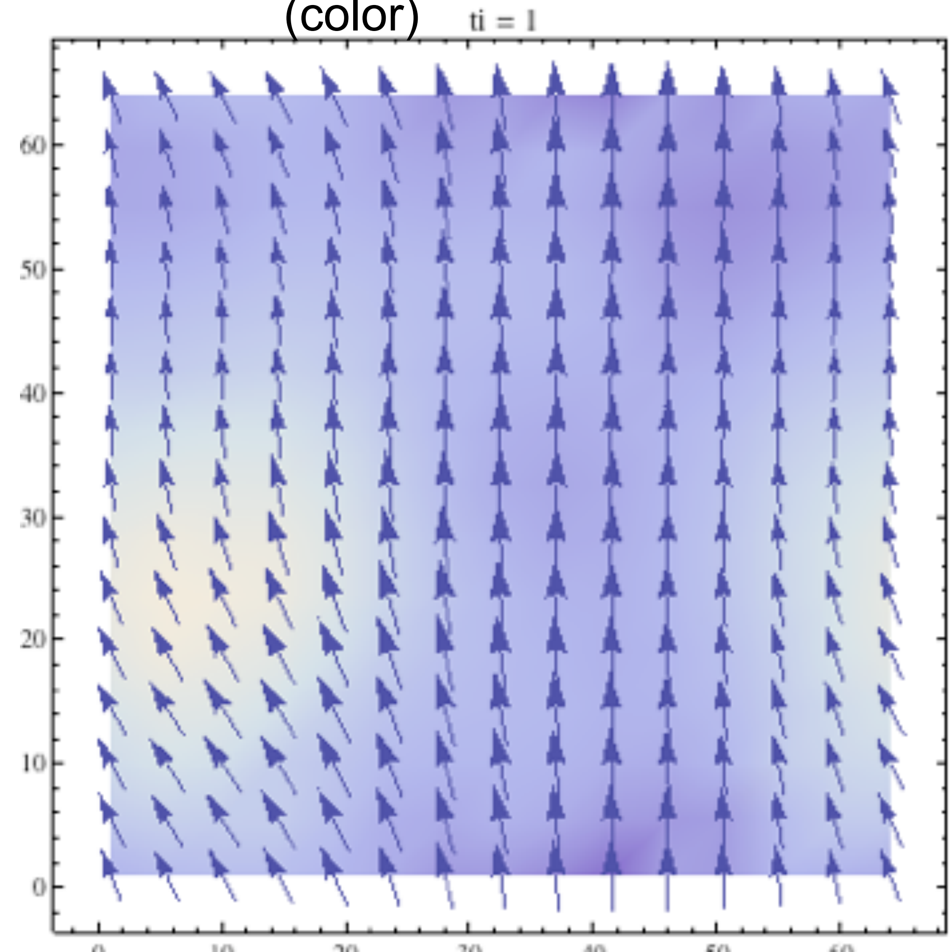
$$G_R^{-1} * u = f_R$$

where: $f_R = \varepsilon^a$

G_R^{-1} is a fractionnal differential operator

ε results from a continuous, vector, multiplicative cascade (Lie cascade)

Complex FIF simulation of a 2D cut of wind and its vorticity (color)



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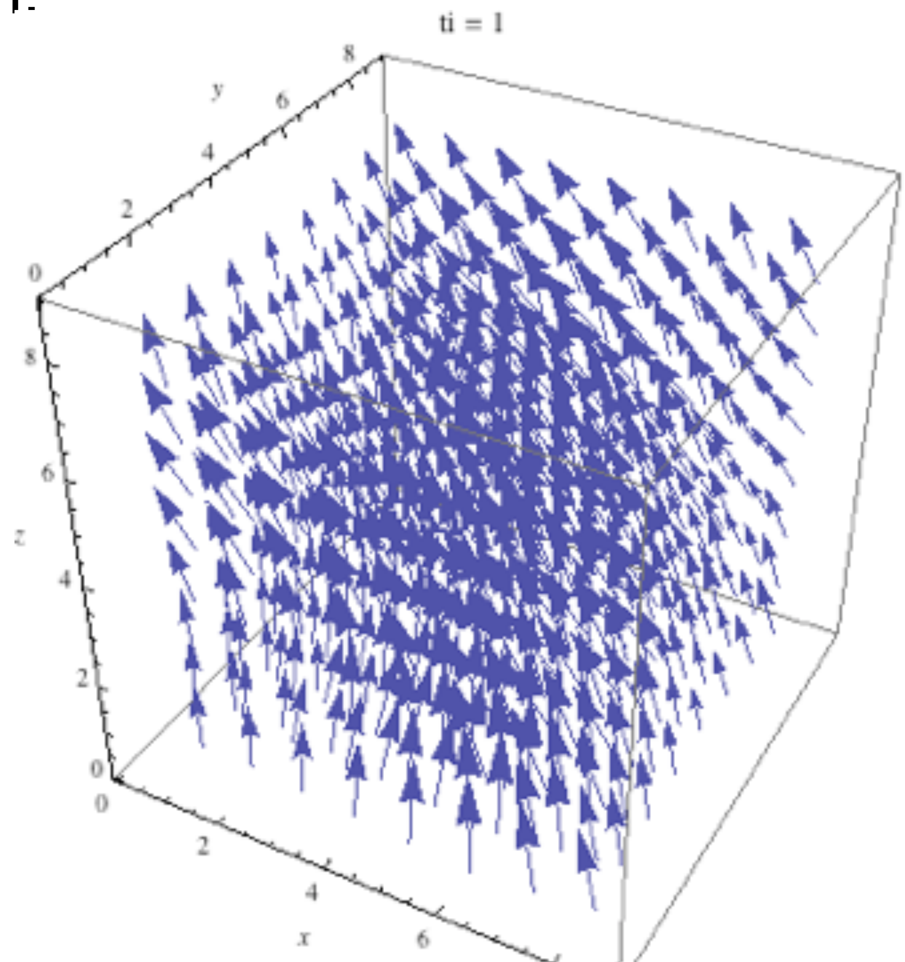
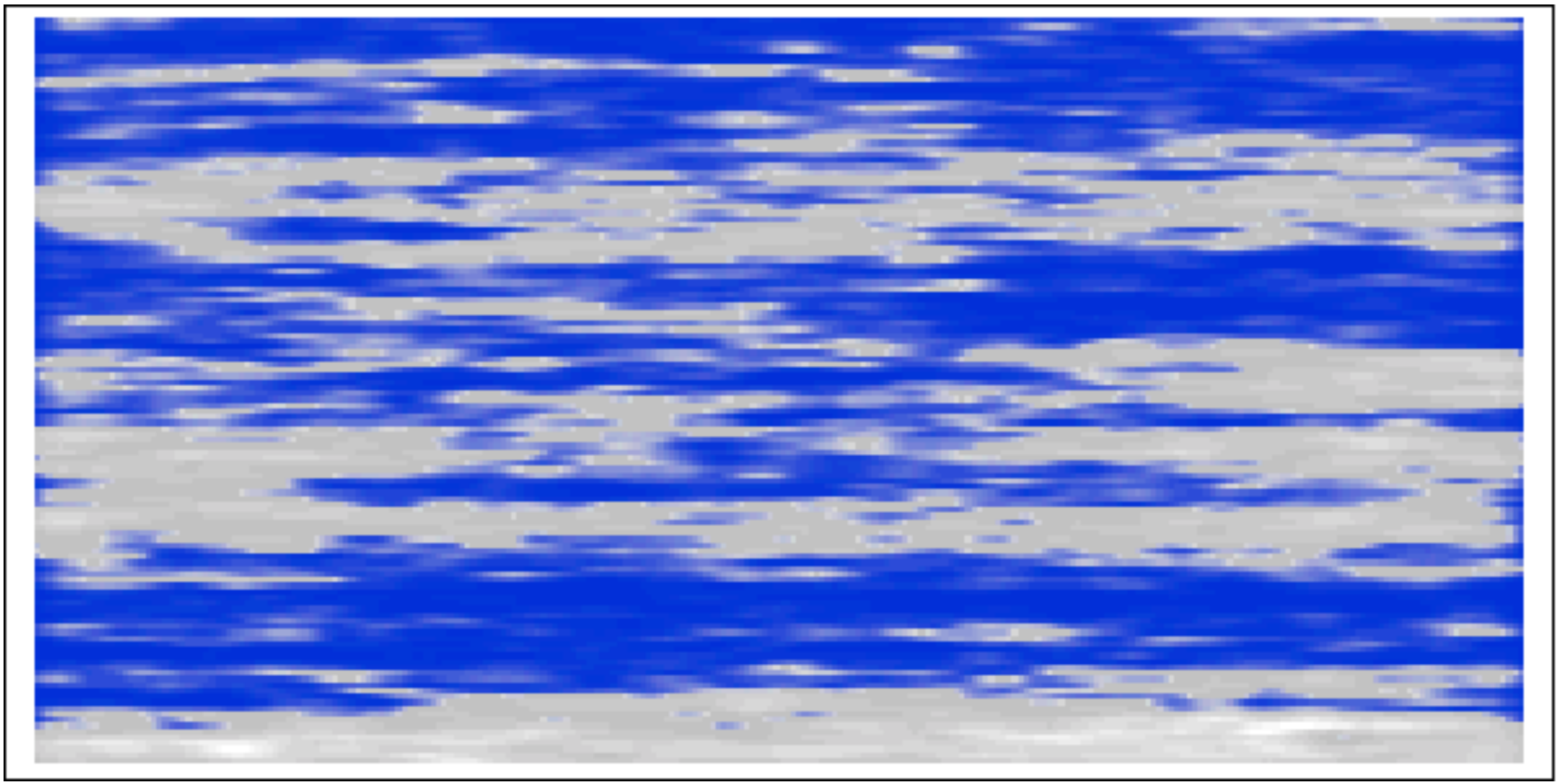
3D FIF wind simulation based on a Clifford algebra



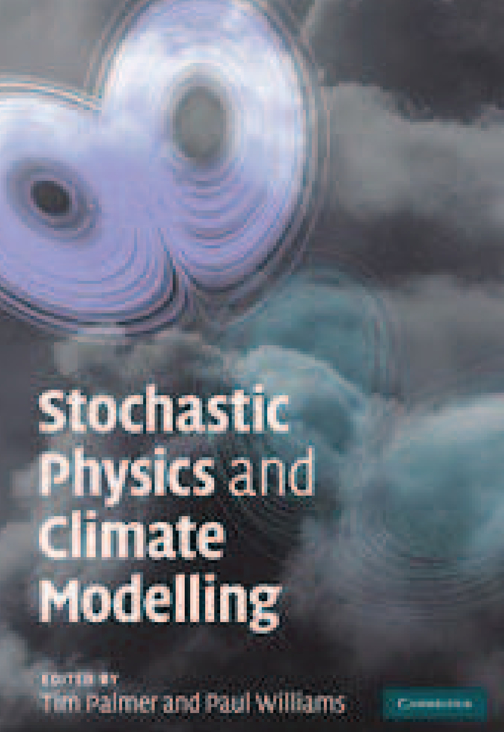
Downscaling of stratified clouds

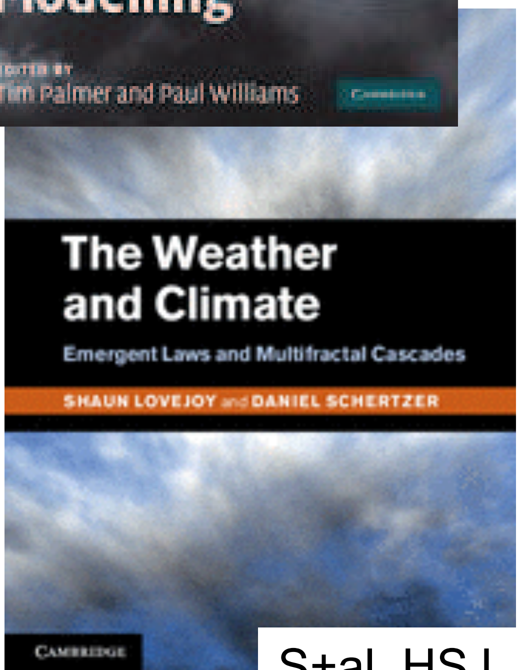
If l_s=1m, then at start: 8km wide, end of zoom: 2.5m wide (factor 32000)



a=1.8, C1=0.05, H=0.33, d=7/9, c=-2/9, e=0., f=0, ls=2^-5 to (about 2^10 at the 108th image (factor1.1 enlargements). There are 4X4 subpixels, 2^7X2^8.

Conclusions





Lindborg +al ACP, 2011

S+al. HSJ. 2010
S+L, IJBC, 2011
S+al. ACP, 2012

- sophisticated numerical simulations of multi-layered QG by Hua and collaborators stimulated a revisit of the QG derivation and the resulting QGT
- scaling analysis is rather opposed to scale analysis (Charney)
  - yields a unique anisotropic scaling regime
  - in fact a (2+*Hz*)-D turbulence ($0 \le Hz \le 1$), with a theoretical $Hz=5/9$
  - generated by a fractional vorticity equation
  - resulting from strongly nonlocal interactions

classical 2-D and 3-D turbulence are not the main options to understand atmospheric and oceanic dynamics

- (2+*Hz*)-D-turbulence:
  - more similarities with 3D, but with distinct features, e.g. pancake structures
  - yields stochastic weather components for climate modeling