

***Ocean Scale Interactions:  
A Tribute to Bach-Lien Hua***

**Brest, 25 June 2014**

# ***Climate Change & Climate Variability: What do We Know?***

**Michael Ghil**

**Ecole Normale Supérieure, Paris, and  
University of California, Los Angeles**

**Based on joint work with A. Bracco, M.D. Chekroun, D. Kondrashov, H. Liu, J.C. McWilliams, J.D. Neelin, E. Simonnet, S. Wang & I. Zaliapin**



**ENS**

***This talk is dedicated to the  
legacy of Lien Hua***



***Please visit these sites for more info.***

**<http://www.atmos.ucla.edu/tcd/>**

**<http://www.environnement.ens.fr/>**

# Lien's legacy

- A life of passion: for science, for her family, & for her colleagues — the young & the young in spirit.
- Rigor & enthusiasm, a broad perspective & a great respect for detail.
- A mastery of the essentials: theory, observations, & high-end modeling.



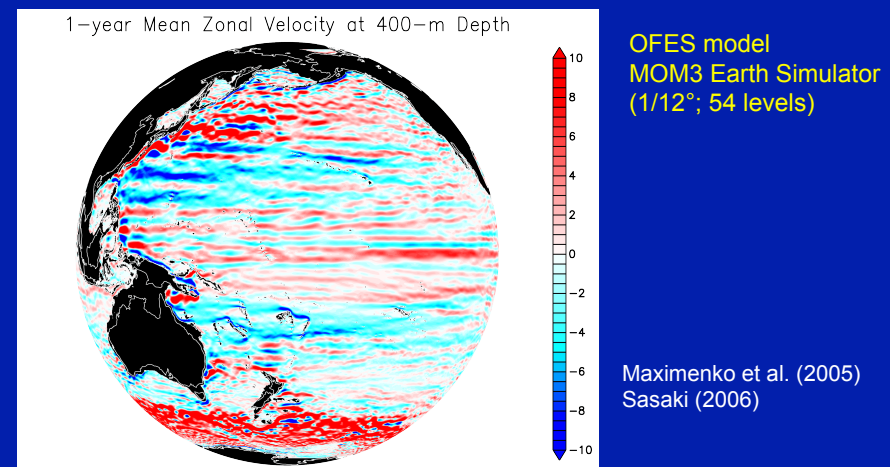
## Nonlinear dynamics of zonal jets in planetary atmospheres and oceans

Bach Lien Hua

Laboratoire de Physique des Océans  
IFREMER, Brest France

## Mid-latitude oceanic flows

- East–West flows **alternating with latitude**;
- quasi-barotropic (surface-intensified) zonal jets.



## **Nonlinear Symmetric Instability and Intraseasonal Oscillations in the Tropical Atmosphere**

**J.-X. ZHAO AND M. GHIL**

*Climate Dynamics Center, Department of Atmospheric Sciences and Institute of Geophysics and Planetary Physics,  
University of California, Los Angeles, California*

(Manuscript received 18 June 1990, in final form 24 March 1991)

### **ABSTRACT**

Symmetric inertial instability (SII) is studied here as a mechanism for stratospheric and tropospheric phenomena in the equatorial regions. We investigate the linear and nonlinear dynamics of SII in a two-layer, zonally symmetric model on an equatorial beta plane, in the presence of a basic flow with horizontal and vertical shear, with and without dissipative effects.

Linear symmetric instabilities are, in accordance with previously published results, purely exponential, that is, nonoscillatory. Nonlinear SII, studied here for the first time on a planetary scale, can produce finite-amplitude oscillatory behavior, periodic or chaotic. The period of oscillations in the inviscid case depends on the initial data. In the presence of dissipative effects, all solutions tend to a limit cycle or to a strange attractor. The dominant period in this case, over a wide range of parameters and whether vertical shear is present or not, is in the intraseasonal, 20–30-day range. It appears therefore that nonlinear SII might be a contributing mechanism to low-frequency oscillations in the tropical atmosphere.

# Motivation

- The *climate system* is highly *nonlinear and* quite *complex*.
- The system's *major components* — the atmosphere, oceans, ice sheets — *evolve* on many time and space scales.
- Its *predictive understanding* has to rely on the system's physical, chemical and biological modeling, but also on the thorough mathematical analysis of the models thus obtained: *the forest vs. the trees*.
- The *hierarchical modeling* approach allows one to give proper weight to the understanding provided by the models vs. their realism: back-and-forth between *“toy”* (conceptual) and *detailed* (“realistic”) *models*, and between *models* and *data*.
- How do we disentangle *natural variability* from *the anthropogenic forcing*: *can we & should we, or not?*

# Climate and Its Sensitivity

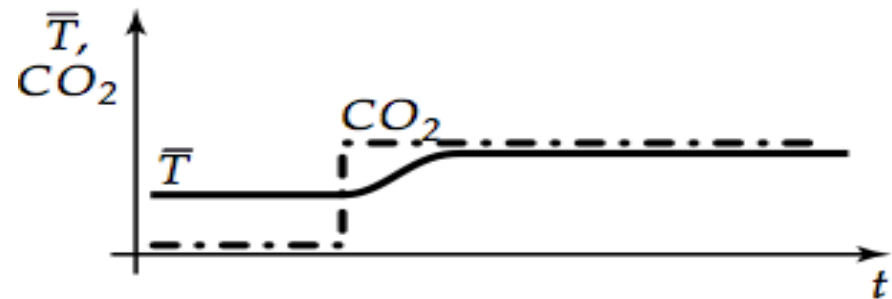
Let's say CO<sub>2</sub> doubles:

How will “climate” change?

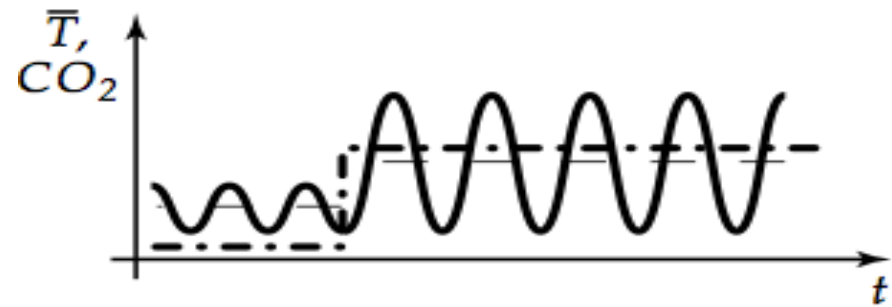
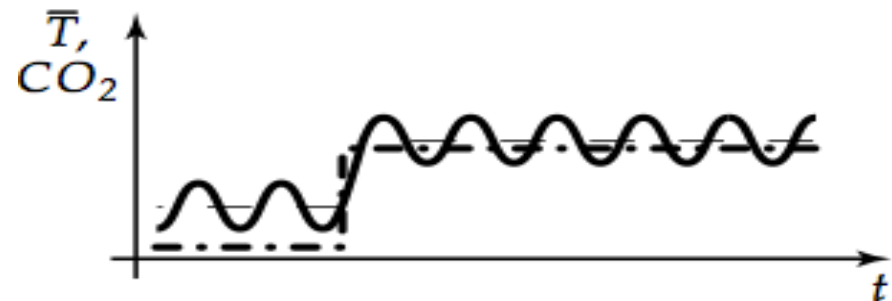
1. Climate is in **stable equilibrium** (fixed point); if so, **mean temperature** will just shift gradually to its new equilibrium value.
2. Climate is **purely periodic**; if so, **mean temperature** will (maybe) shift gradually to its new equilibrium value. But how will the **period, amplitude and phase** of the **limit cycle** change?
3. And how about some “real stuff” now: **chaotic + random**?

Ghil (in *Encycl. Global Environmental Change*, 2002)

a) *Equilibrium sensitivity*



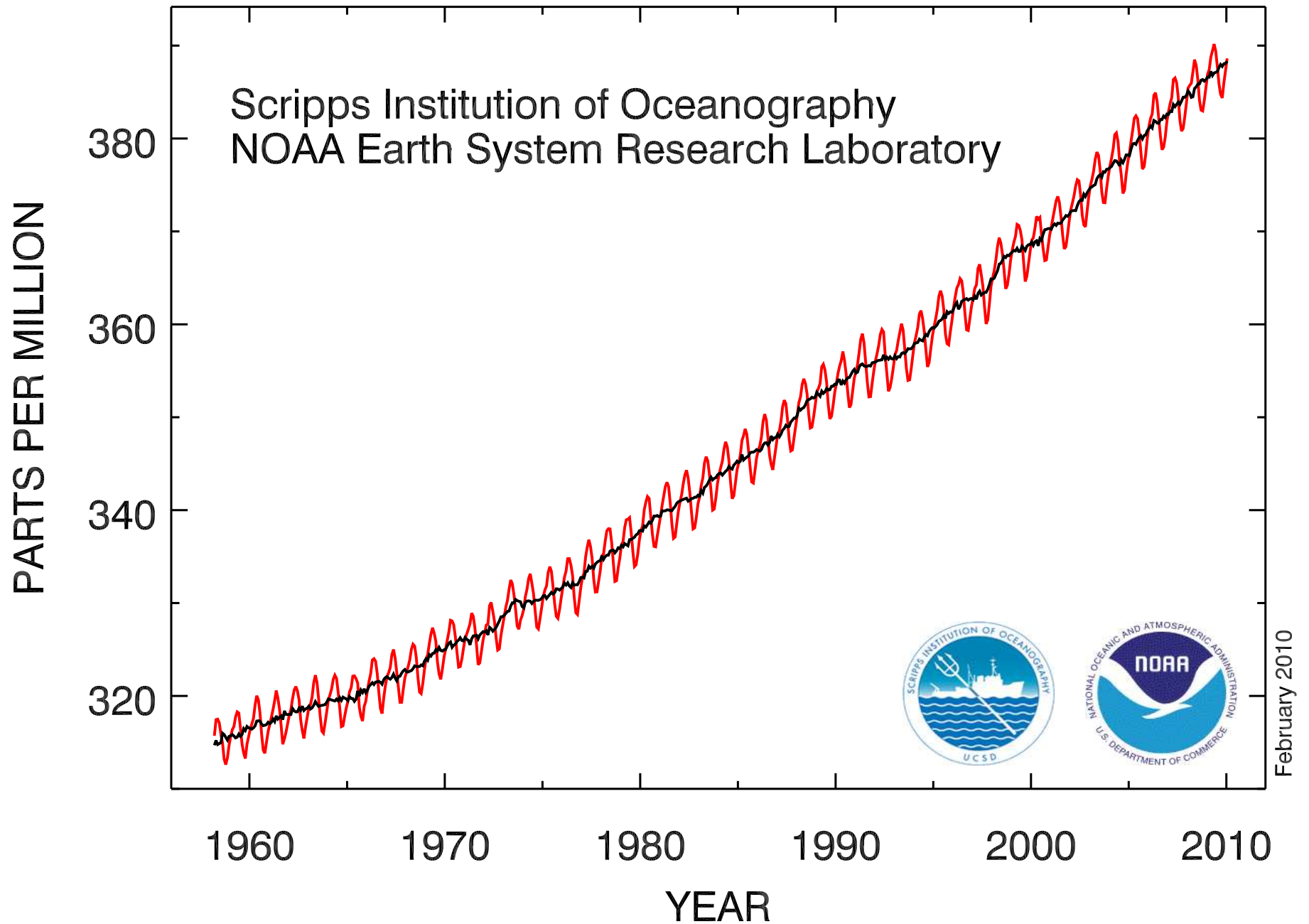
b) *Nonequilibrium sensitivity*



# Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data → error growth
  - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Linear response theory and climate sensitivity
- Conclusions and references
  - natural variability and anthropogenic forcing: the “grand unification”
  - selected bibliography

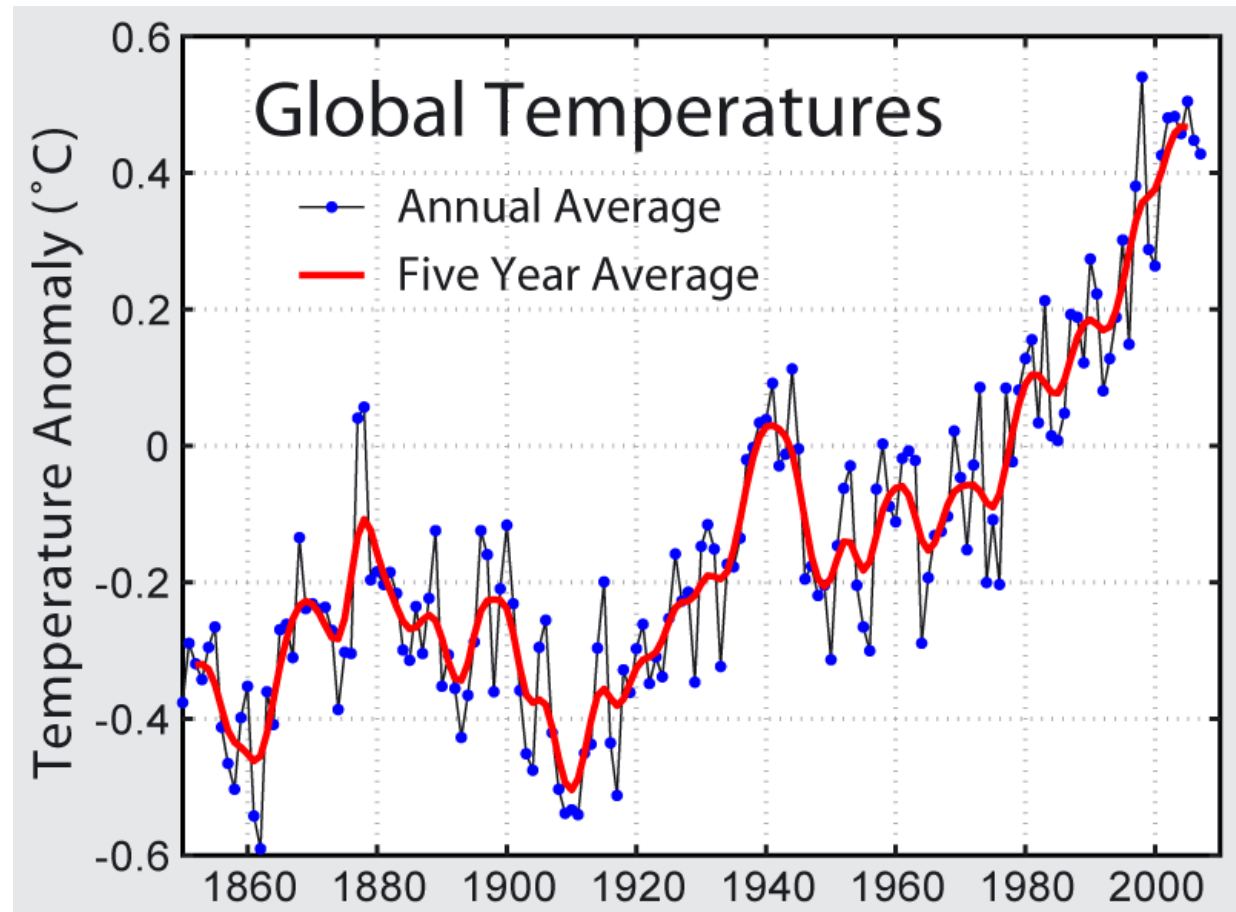
# Atmospheric CO<sub>2</sub> at Mauna Loa Observatory



# Temperatures and GHGs

Greenhouse gases (GHGs) go up,  
temperatures go up:

It's gotta do with us, at least a bit,  
doesn't it?



Wikicommons, from  
Hansen *et al.* (PNAS, 2006);  
see also <http://data.giss.nasa.gov/gistemp/graphs/>

Unfortunately, things aren't all that easy!

*What to do?*

*Try to achieve better interpretation of, and agreement between, models ...*

Ghil, M., 2002: Natural climate variability, in *Encyclopedia of Global Environmental Change*, T. Munn (Ed.), Vol. 1, Wiley

Natural variability introduces additional complexity into the anthropogenic climate change problem

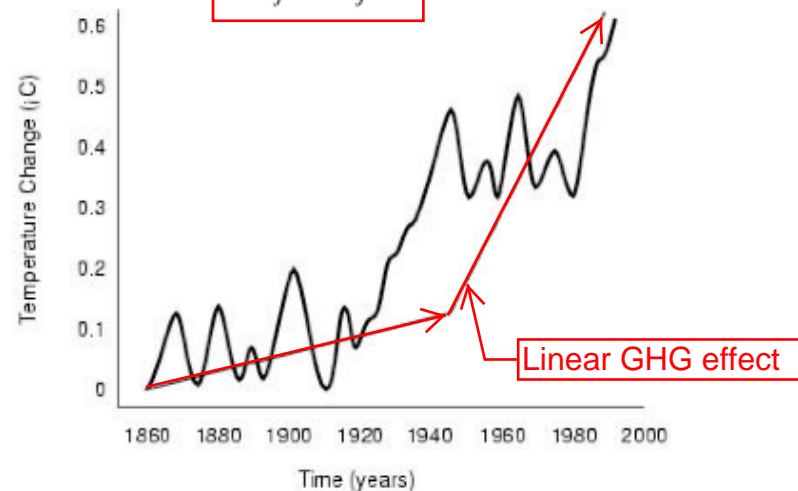
The most common interpretation of observations and GCM simulations of climate change is still in terms of a scalar, linear Ordinary Differential Equation (ODE)

$$c \frac{dT}{dt} = -kT + Q$$

$k = \sum k_i$  – feedbacks (+ve and -ve)

$Q = \sum Q_j$  – sources & sinks

$$Q_j = Q_j(t)$$



Linear response to CO<sub>2</sub> vs. observed change in T

Hence, we need to consider instead a system of nonlinear Partial Differential Equations (PDEs), with parameters and multiplicative, as well as additive forcing (deterministic + stochastic)

$$\frac{dX}{dt} = N(X, t, \mu, \beta)$$

# Global warming and its socio-economic impacts

Temperatures rise:

- What about impacts?
- How to adapt?

The answer, my friend, is blowing in the wind, *i.e.*, it depends on the accuracy and reliability of the forecast ...

*Source : IPCC (2007), AR4, WGI, SPM*

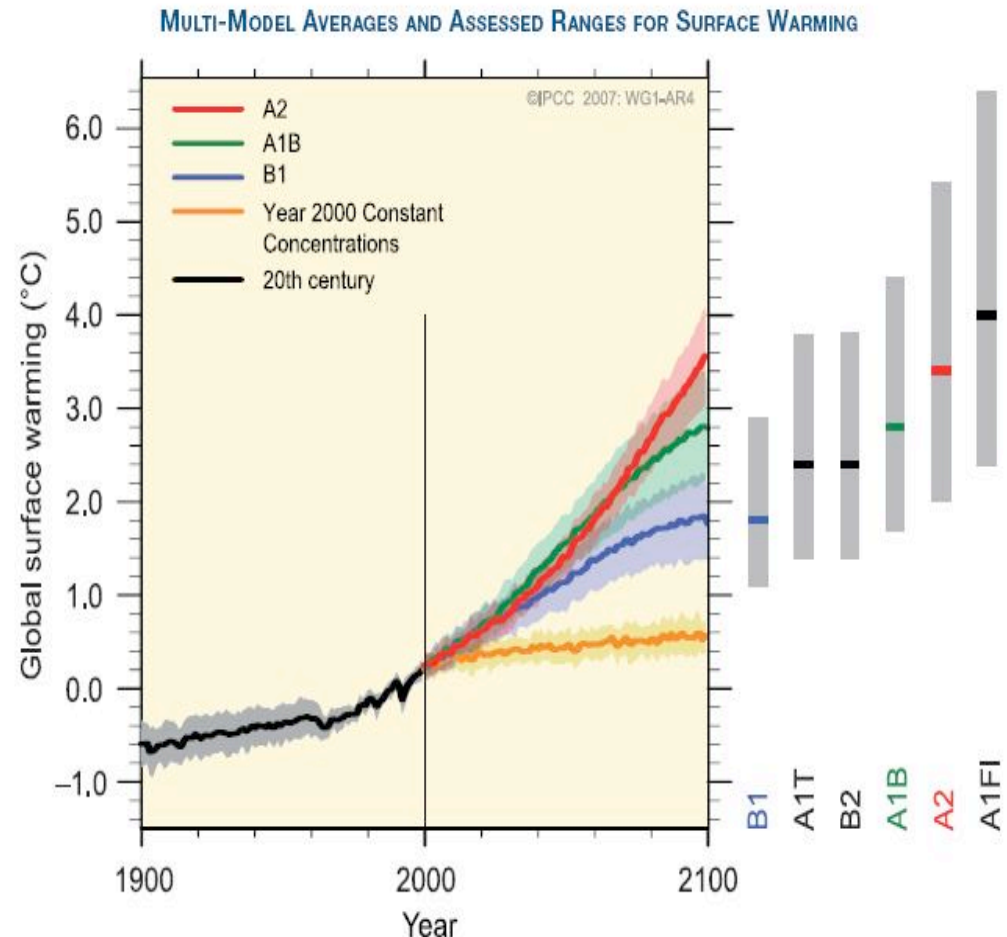


Figure SPM.5. Solid lines are multi-model global averages of surface warming (relative to 1980–1999) for the scenarios A2, A1B and B1, shown as continuations of the 20th century simulations. Shading denotes the  $\pm 1$  standard deviation range of individual model annual averages. The orange line is for the experiment where concentrations were held constant at year 2000 values. The grey bars at right indicate the best estimate (solid line within each bar) and the likely range assessed for the six SRES marker scenarios. The assessment of the best estimate and likely ranges in the grey bars includes the AOGCMs in the left part of the figure, as well as results from a hierarchy of independent models and observational constraints. (Figures 10.4 and 10.29)

# Global warming and its socio-economic impacts– II

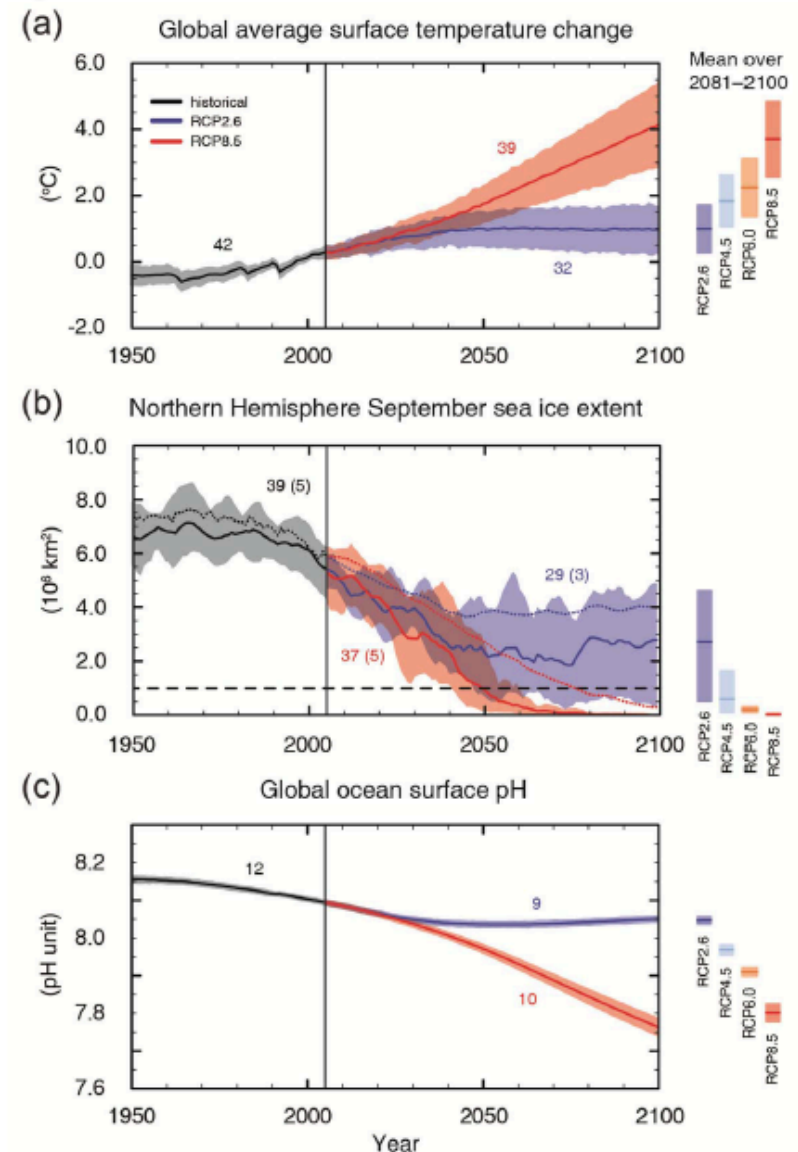
Temperatures rise:

- What about impacts?
- How to adapt?

## AR5 vs. AR4

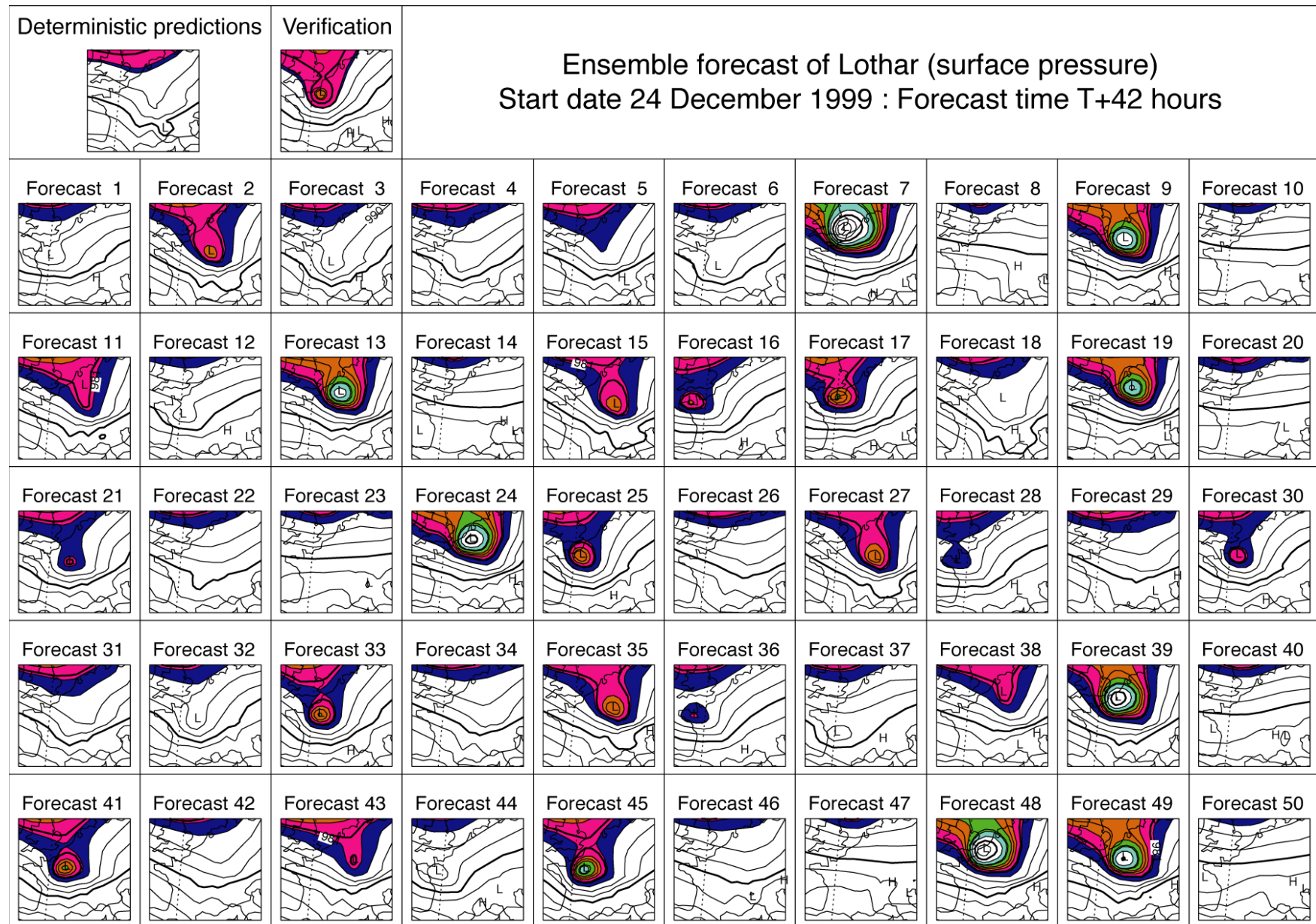
A certain air of *déjà vu*:  
GHG “scenarios” have been replaced by “representative concentration pathways” (RCPs), more dire predictions, but the **uncertainties** remain.

Source : IPCC (2013),  
AR5, WGI, SPM



# Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data → error growth
  - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Linear response theory and climate sensitivity
- Conclusions and references
  - natural variability and anthropogenic forcing: the “grand unification”
  - selected bibliography



Courtesy Tim Palmer, 2009

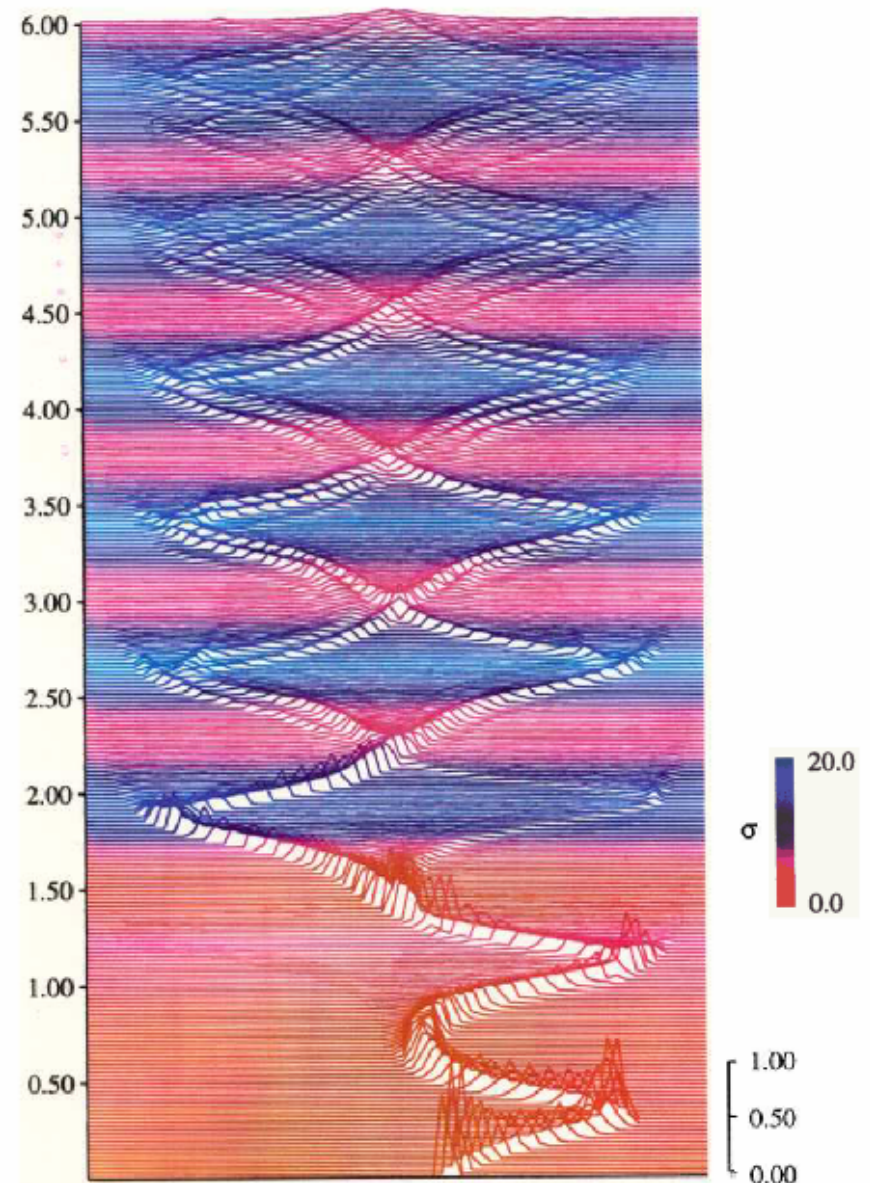
# Exponential divergence vs. “coarse graining”

The classical view of dynamical systems theory is:  
positive Lyapunov exponent →  
trajectories diverge exponentially

But the presence of multiple regimes implies a much more structured behavior in phase space

Still, the probability distribution function (pdf), when calculated forward in time, is pretty smeared out

L. A. Smith (*Encycl. Atmos. Sci.*, 2003)



# So what's it gonna be like, by 2100?

**Table SPM.2.** Recent trends, assessment of human influence on the trend and projections for extreme weather events for which there is an observed late-20th century trend. (Tables 3.7, 3.8, 9.4; Sections 3.8, 5.5, 9.7, 11.2–11.9)

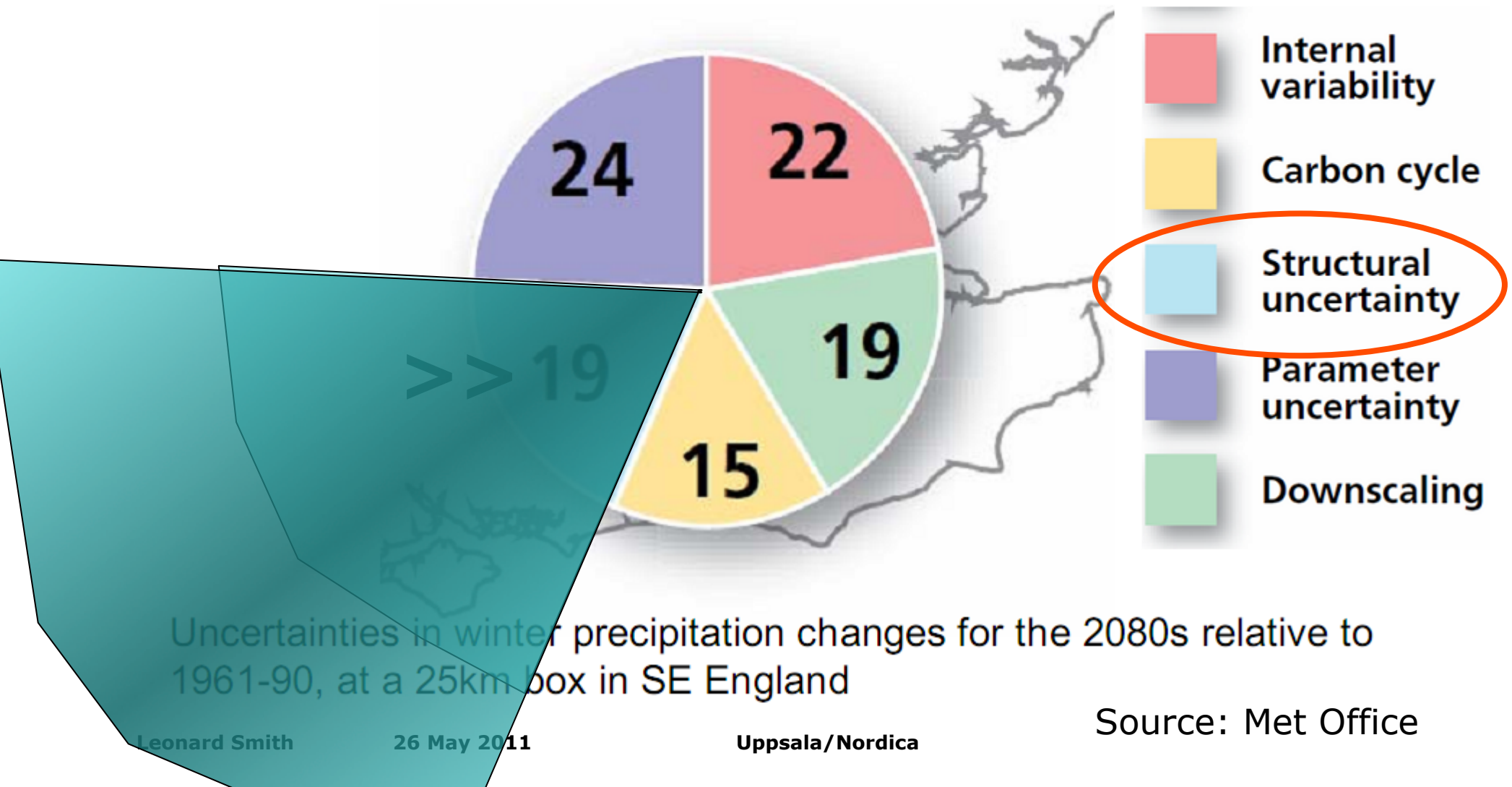
Phenomenon <sup>a</sup> and direction of trend	Likelihood that trend occurred in late 20th century (typically post 1960)	Likelihood of a human contribution to observed trend <sup>b</sup>	Likelihood of future trends based on projections for 21st century using SRES scenarios
Warmer and fewer cold days and nights over most land areas	<i>Very likely<sup>c</sup></i>	<i>Likely<sup>d</sup></i>	<i>Virtually certain<sup>d</sup></i>
Warmer and more frequent hot days and nights over most land areas	<i>Very likely<sup>e</sup></i>	<i>Likely (nights)<sup>d</sup></i>	<i>Virtually certain<sup>d</sup></i>
Warm spells/heat waves. Frequency increases over most land areas	<i>Likely</i>	<i>More likely than not<sup>f</sup></i>	<i>Very likely</i>
Heavy precipitation events. Frequency (or proportion of total rainfall from heavy falls) increases over most areas	<i>Likely</i>	<i>More likely than not<sup>f</sup></i>	<i>Very likely</i>
Area affected by droughts increases	<i>Likely in many regions since 1970s</i>	<i>More likely than not</i>	<i>Likely</i>
Intense tropical cyclone activity increases	<i>Likely in some regions since 1970</i>	<i>More likely than not<sup>f</sup></i>	<i>Likely</i>
Increased incidence of extreme high sea level (excludes tsunamis) <sup>g</sup>	<i>Likely</i>	<i>More likely than not<sup>h</sup></i>	<i>Likely<sup>i</sup></i>

# Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data → error growth
  - sensitivity to model formulation → see below!
- **Uncertainties and how to fix them**
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Linear response theory and climate sensitivity
- Conclusions and references
  - natural variability and anthropogenic forcing: the “grand unification”
  - selected bibliography

# How important are different sources of uncertainty?

- Varies, but typically no single source dominates.



# Can we, nonlinear dynamicists, help?

The uncertainties  
might be *intrinsic*,  
rather than mere  
“tuning problems”

If so, maybe  
*stochastic structural  
stability* could help!

Might fit in nicely with  
recent taste for  
“stochastic  
parameterizations”

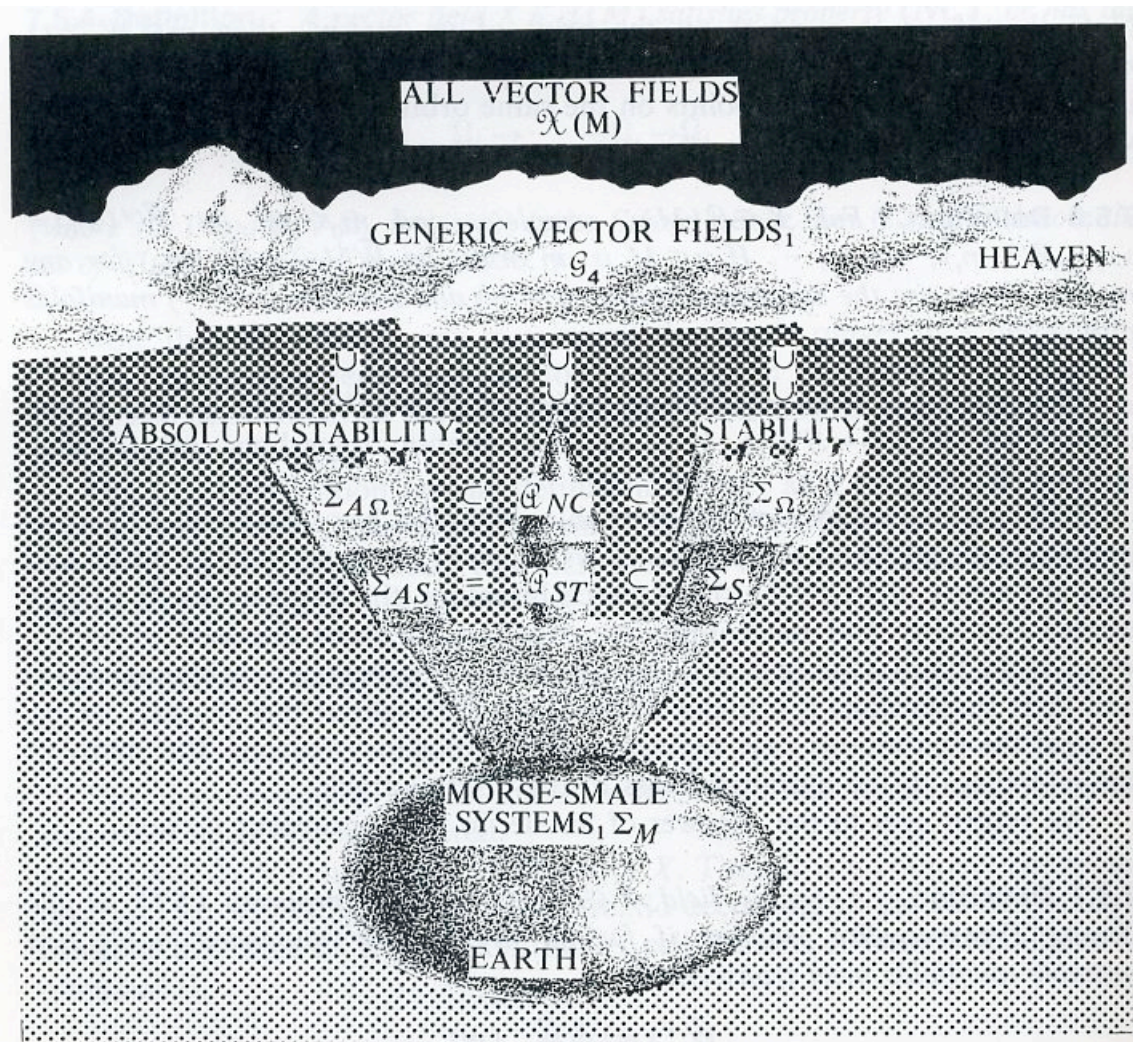


Figure 7.5-1. The three towers of differentiable dynamics.

*The DDS dream of structural stability* (from Abraham & Marsden, 1978)

# Non-autonomous Dynamical Systems

## A linear, dissipative, forced example: *forward vs. pullback attraction*

Consider the scalar, linear ordinary differential equation (ODE)

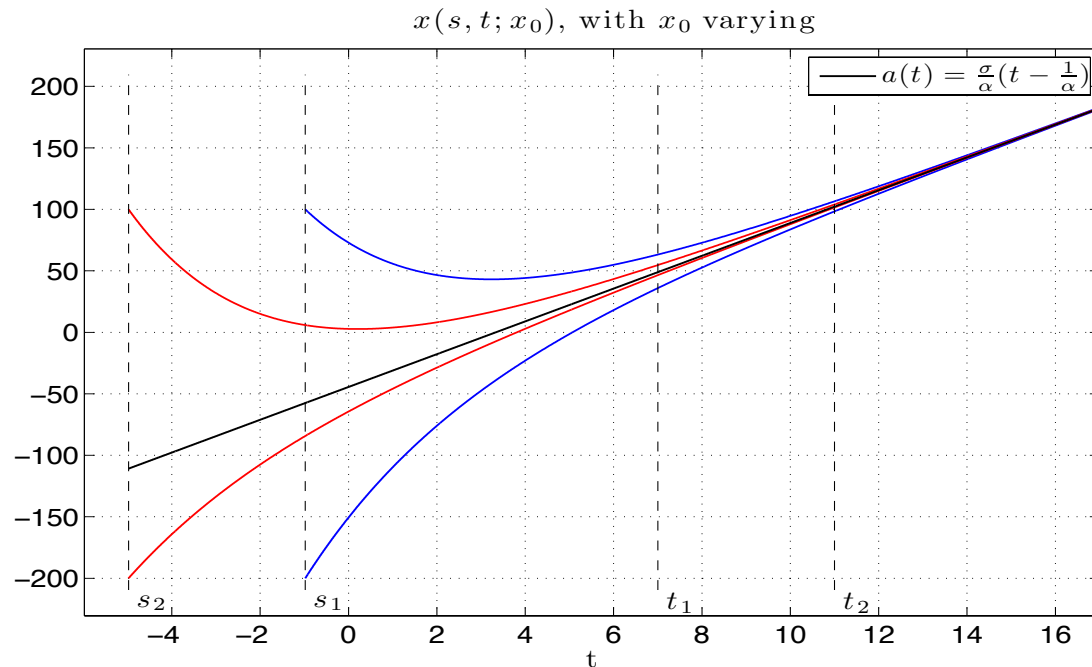
$$\dot{x} = -\alpha x + \sigma t, \quad \alpha > 0, \quad \sigma > 0.$$

The autonomous part of this ODE,  $\dot{x} = -\alpha x$ , is **dissipative** and all solutions  $x(t; x_0) = x(t; x(0) = x_0)$  converge to 0 as  $t \rightarrow +\infty$ .

What about the non-autonomous, forced ODE? As the energy being put into the system by the forcing is dissipated, we expect things to change in time. In fact, if we “pull back” far enough, replace  $x(t; x_0)$  by  $x(s, t; x_0) = x(s, t; x(s) = x_0)$ ,

and let  $s \rightarrow -\infty$ , we get the **pullback attractor**  $a = a(t)$  in the figure,

$$a(t) = \frac{\sigma}{\alpha} \left( t - \frac{1}{\alpha} \right).$$

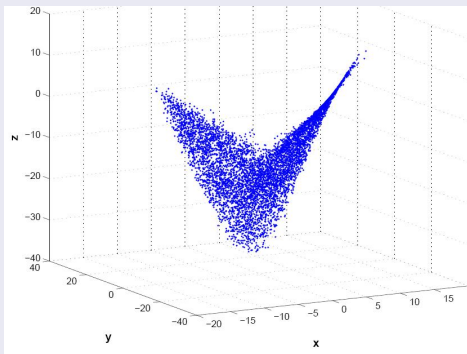


# Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data → error growth
  - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Linear response theory and climate sensitivity
- Conclusions and references
  - natural variability and anthropogenic forcing: the “grand unification”
  - selected bibliography

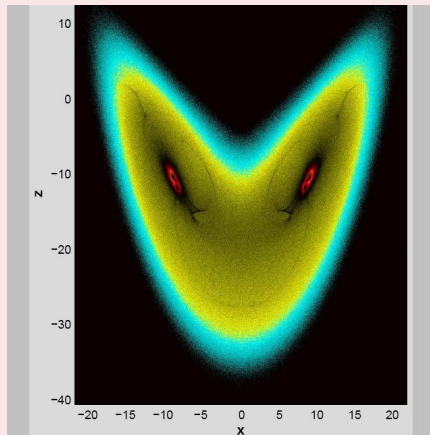
# Random attractor of the stochastic Lorenz system

## Snapshot of the random attractor (RA)



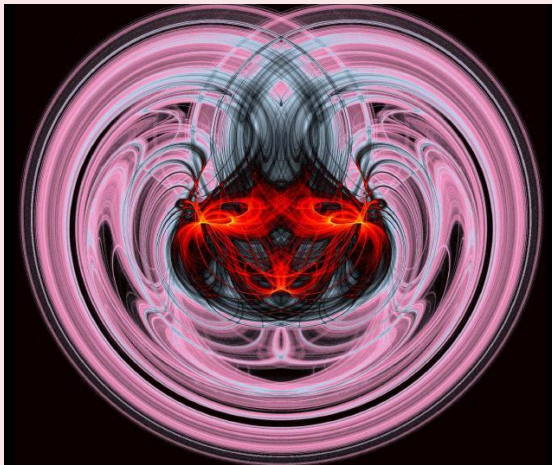
- A **snapshot** of the RA,  $\mathcal{A}(\omega)$ , computed at a fixed time  $t$  and for the **same realization**  $\omega$ ; it is made up of points transported by the stochastic flow, from the remote past  $t - T$ ,  $T \gg 1$ .
- We use **multiplicative noise** in the deterministic Lorenz model, with the classical parameter values  $b = 8/3$ ,  $\sigma = 10$ , and  $r = 28$ .
- Even computed **pathwise**, this object supports meaningful **statistics**.

# Sample measures supported by the R.A.



- We compute the probability measure on the R.A. at some fixed time  $t$ , and for a fixed realization  $\omega$ . We show a “projection”,  $\int \mu_\omega(x, y, z) dy$ , with **multiplicative noise**:  $dx_i = \text{Lorenz}(x_1, x_2, x_3)dt + \alpha x_i dW_t; i \in \{1, 2, 3\}$ .
- **10 million of initial points** have been used for this picture!

# Sample measure supported by the R.A.



- Still 1 Billion I.D., and  $\alpha = 0.5$ . Another one?

# Sample measure supported by the R.A.

## Sample measures evolve with time.

- Recall that these sample measures are the **frozen statistics** at a time  $t$  for a realization  $\omega$ .
- How do these **frozen statistics** evolve with time?
- **Action!**



A day in the life of the Lorenz (1963) model's random attractor, or LORA for short;  
see SI in Chekroun, Simonnet & Ghil (2011, *Physica D*)

# Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data → error growth
  - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- **Linear response theory and climate sensitivity**
- Conclusions and references
  - natural variability and anthropogenic forcing: the “grand unification”
  - selected bibliography

# Climate and Its Sensitivity

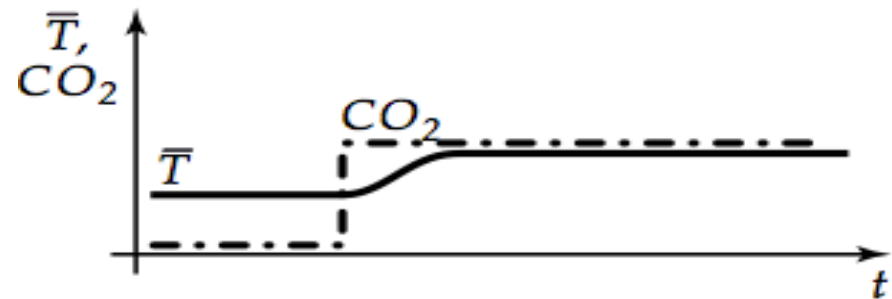
Let's say CO<sub>2</sub> doubles:

How will “climate” change?

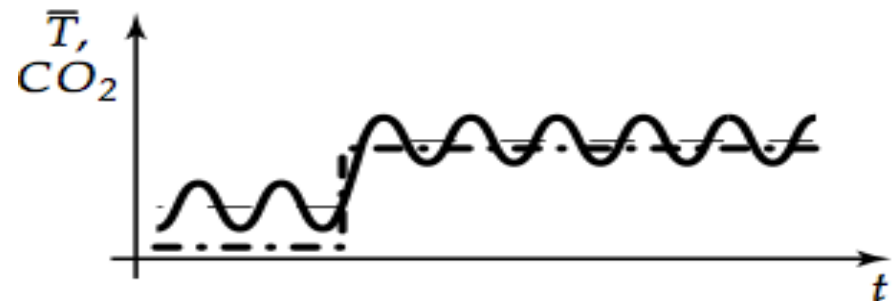
1. Climate is in **stable equilibrium** (fixed point); if so, **mean temperature** will just shift gradually to its new equilibrium value.
2. Climate is **purely periodic**; if so, **mean temperature** will (maybe) shift gradually to its new equilibrium value. But how will the **period, amplitude and phase** of the **limit cycle** change?
3. And how about some “real stuff” now: **chaotic + random**?

Ghil (Encycl. Global Environmental Change, 2002)

a) *Equilibrium sensitivity*



b) *Nonequilibrium sensitivity*

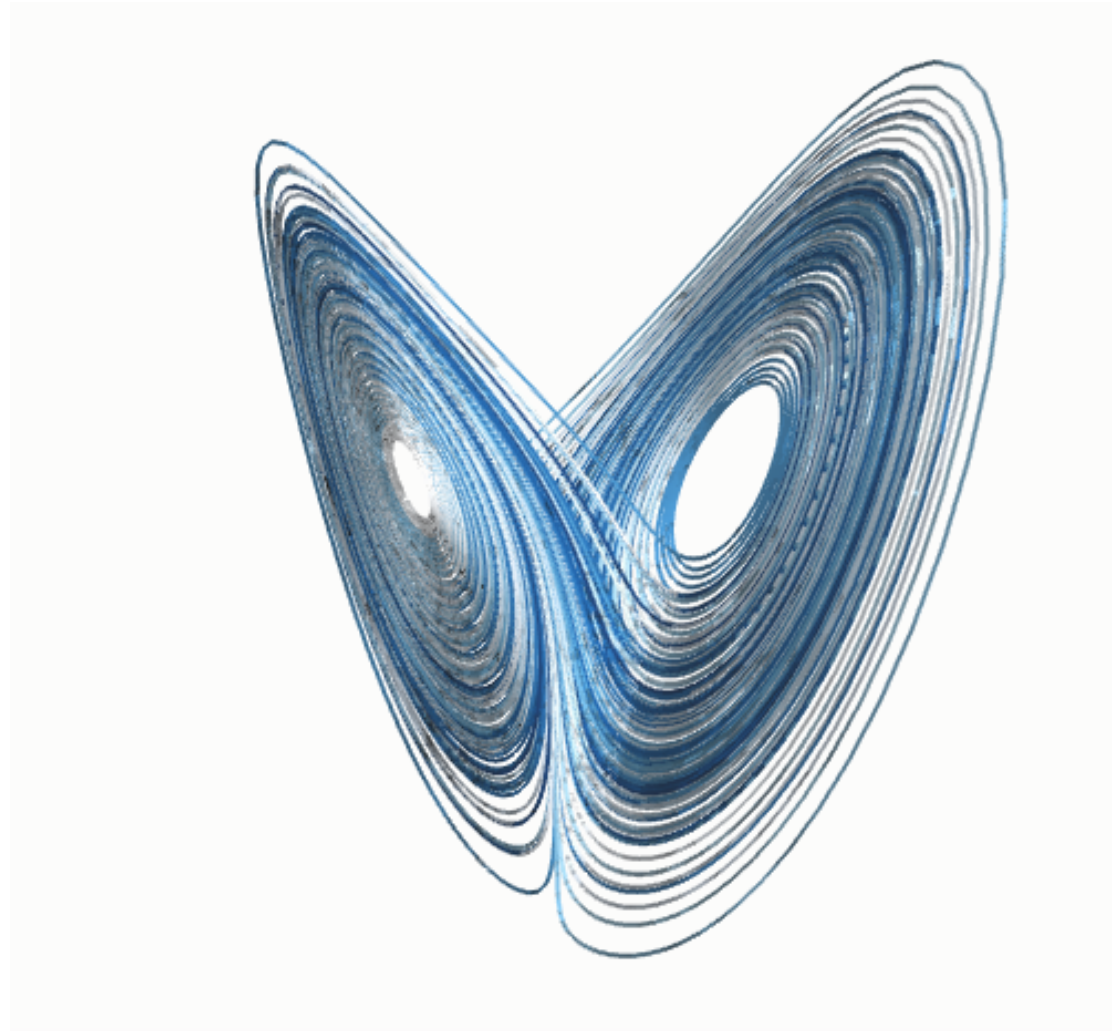


# Classical Strange Attractor

Physically **closed** system, modeled mathematically as **autonomous** system: neither deterministic (anthropogenic) nor random (natural) forcing.

The **attractor** is **strange**, but still fixed in time  $\sim$  “**irrational**” number.

**Climate sensitivity**  $\sim$  change in the **average value (first moment)** of the coordinates  $(x, y, z)$  as a **parameter  $\lambda$**  changes.



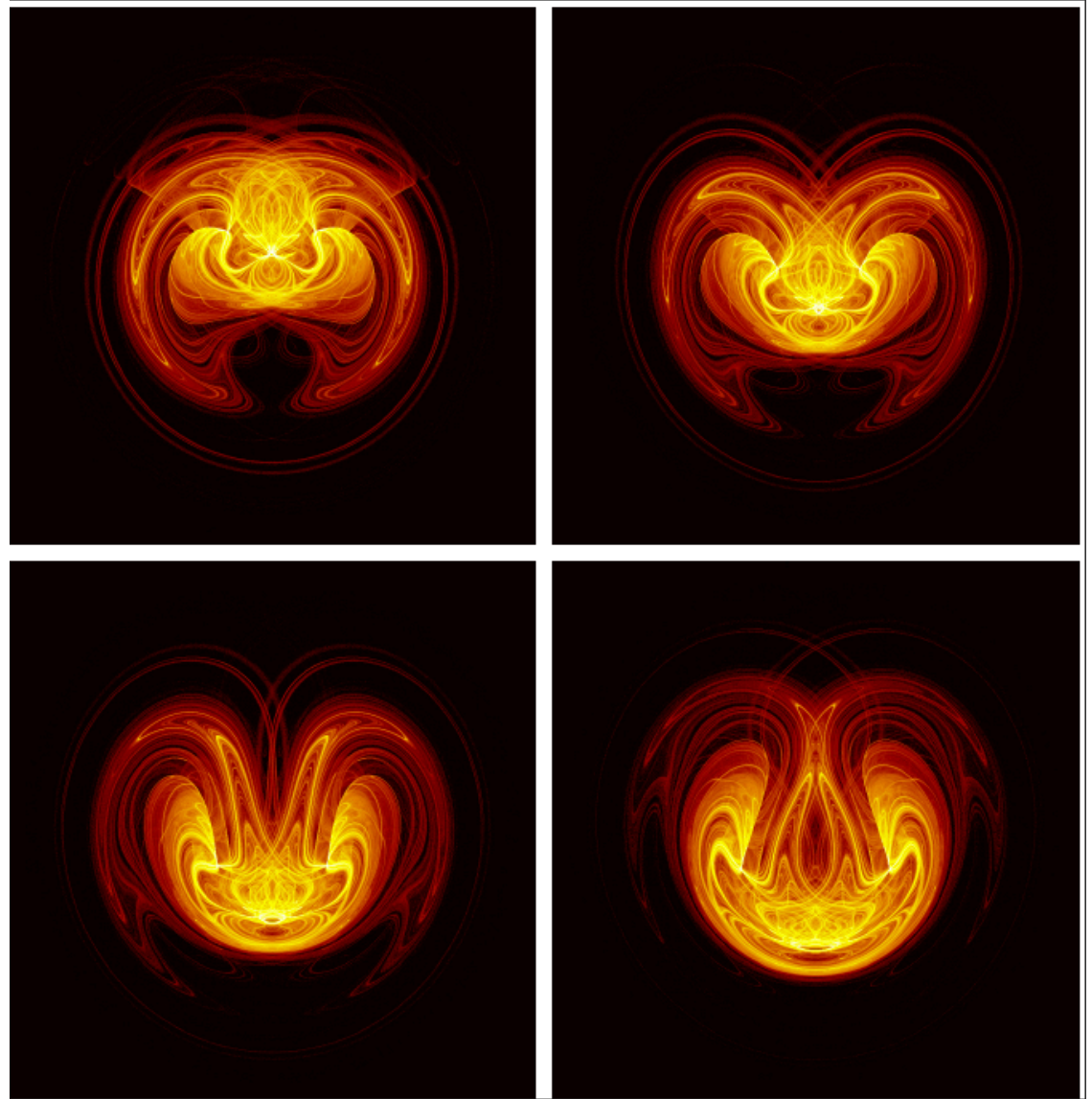
# Random Attractor

Physically **open** system, modeled mathematically as **non-autonomous** system: allows for deterministic (anthropogenic) as well as random (natural) forcing.

The **attractor** is “**pullback**” and evolves in time ~ “**imaginary**” or “**complex**” number.

**Climate sensitivity** ~ change in the statistical properties (first and **higher-order moments**) of the **attractor** as one or more **parameters** ( $\lambda$ ,  $\mu$ , ...) change.

Ghil (*Encyclopedia of Atmospheric Sciences*, 2<sup>nd</sup> ed., 2012)



# Parameter dependence – I

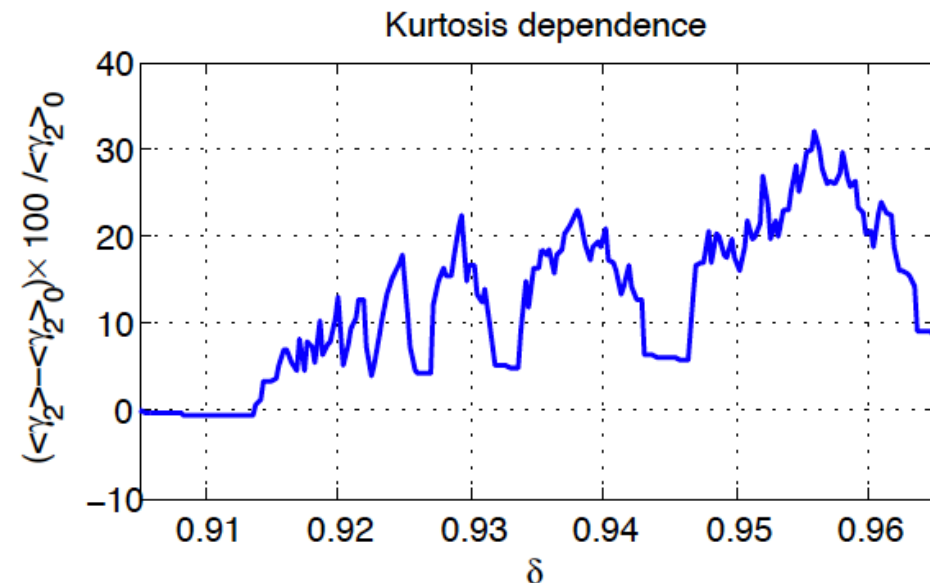
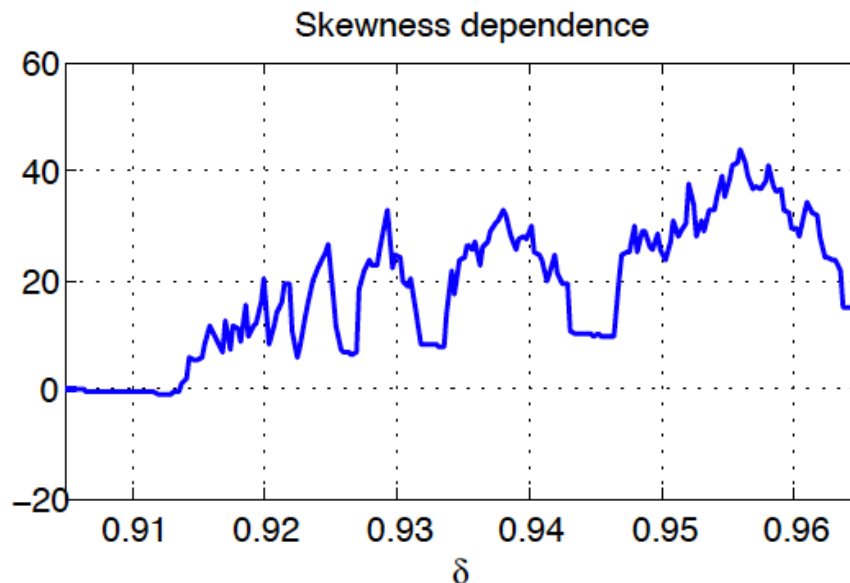
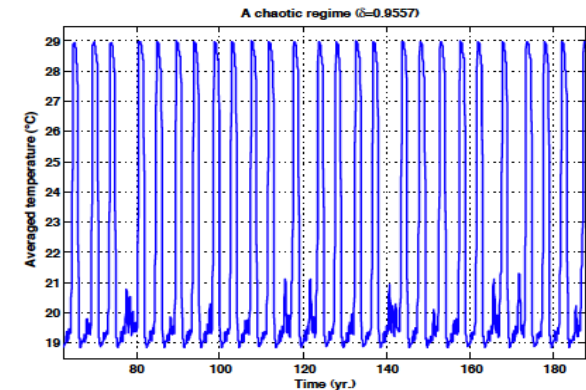
$$\delta = 0.9557$$

It can be smooth or it can be rough:

Niño-3 SSTs from intermediate coupled model for ENSO (Jin, Neelin & Ghil, 1994, 1996)

Skewness & kurtosis of the SSTs:  
time series of 4000 years,

$$\Delta\delta = 3 \cdot 10^{-4}$$



M. Chekroun & D. Kondrashov (work in progress)

# Sample measures for an NDDE model of ENSO

## The Galanti-Tziperman (GT) model (JAS, 1999)

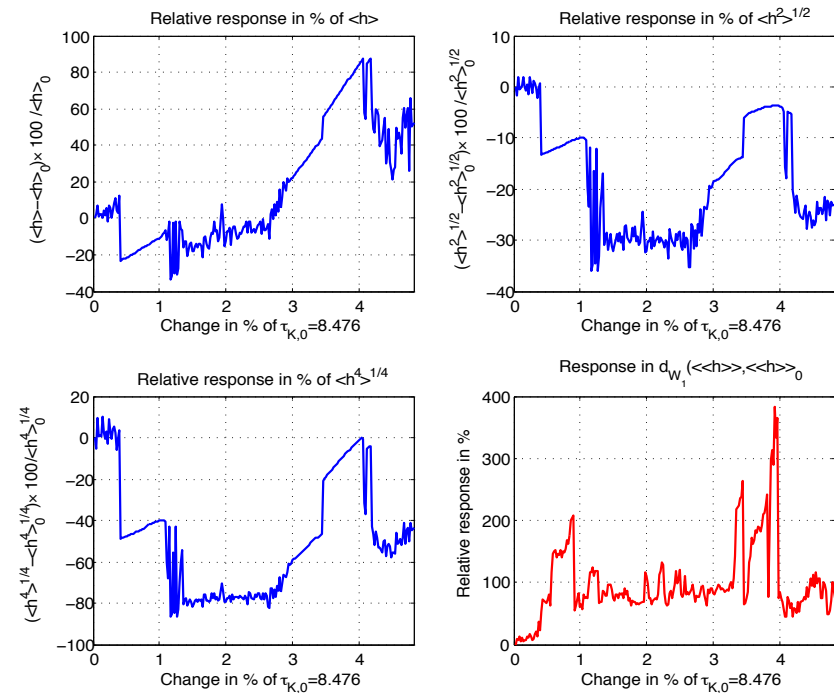
$$\frac{dT}{dt} = -\epsilon_T T(t) - M_0(T(t) - T_{sub}(h(t))),$$

Neutral delay-differential equation (NDDE),  
derived from Cane-Zebiak and Jin-Neelin  
models for ENSO:  $T$  is East-basin SST  
and  $h$  is thermocline depth.

$$h(t) = M_1 e^{-\epsilon_m(\tau_1 + \tau_2)} h(t - \tau_1 - \tau_2) \\ - M_2 \tau_1 e^{-\epsilon_m(\frac{\tau_1}{2} + \tau_2)} \mu(t - \tau_2 - \frac{\tau_1}{2}) T(t - \tau_2 - \frac{\tau_1}{2}) \\ + M_3 \tau_2 e^{-\epsilon_m \frac{\tau_2}{2}} \mu(t - \frac{\tau_2}{2}) T(t - \frac{\tau_2}{2}).$$

Seasonal forcing given by  
 $\mu(t) = 1 + \epsilon \cos(\omega t + \phi)$ .  
The pullback attractor and its  
invariant measures were computed.

Figure shows the changes in the mean,  
2<sup>nd</sup> & 4<sup>th</sup> moment of  $h(t)$ , along with the  
Wasserstein distance  $d_W$ , for changes  
of 0–5% in the delay parameter  $\tau_{K,0}$ .



Note intervals of both **smooth** & **rough** dependence!

# Pullback attractor and invariant measure of the GT model

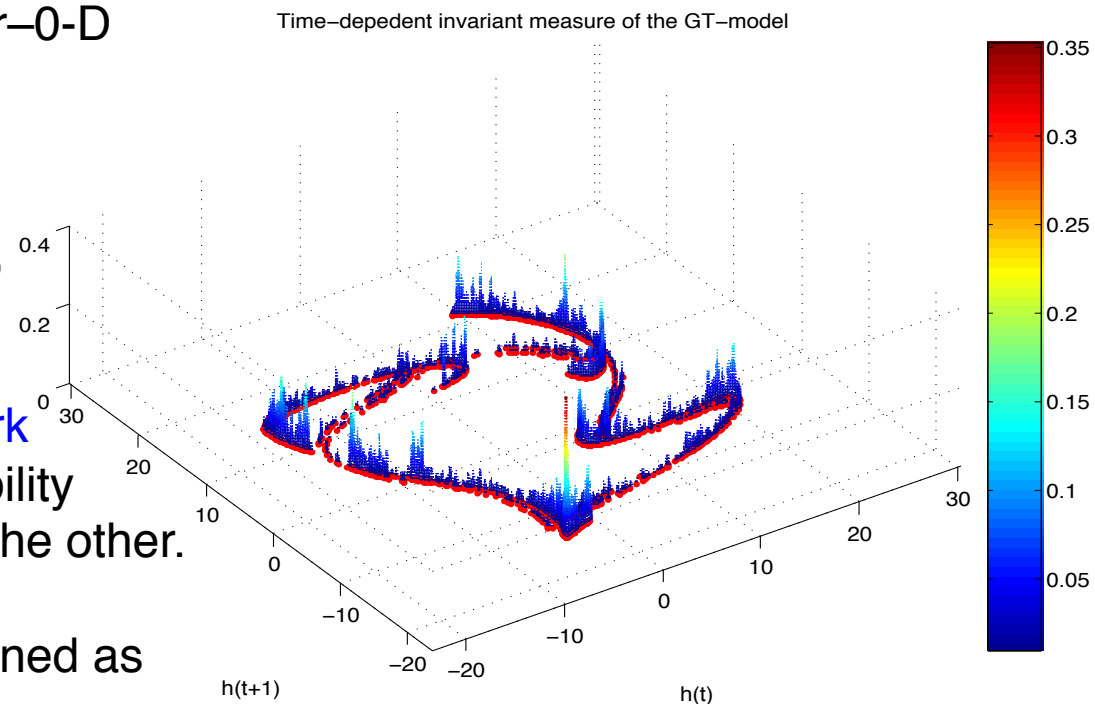
The time-dependent pullback attractor of the GT model supports an invariant measure  $\nu = \nu(t)$ , whose density is plotted in 3-D perspective.

The plot is in delay coordinates  $h(t+1)$  vs.  $h(t)$  and the density is highly concentrated along 1-D filaments and, furthermore, exhibits sharp, near-0-D peaks on these filaments.

The Wasserstein distance  $d_W$  between one such configuration, at given parameter values, and another one, at a different set of values, is proportional to the work needed to move the total probability mass from one configuration to the other.

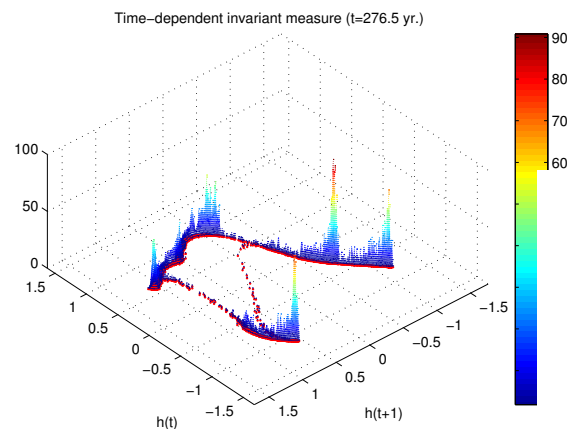
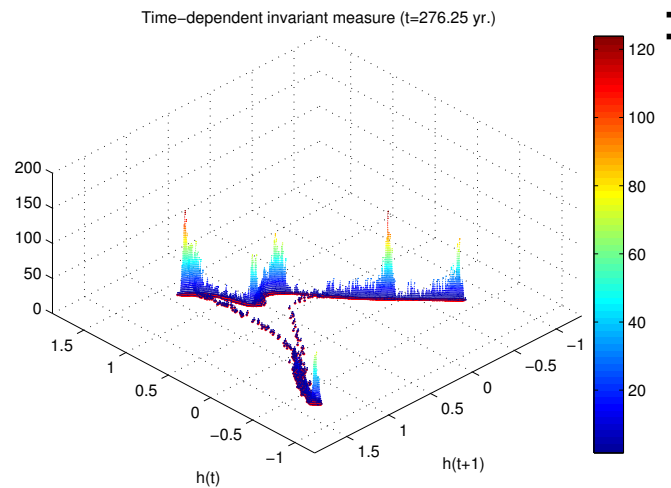
Climate sensitivity  $\gamma$  can be defined as

$$\gamma = \partial d_W / \partial \tau$$

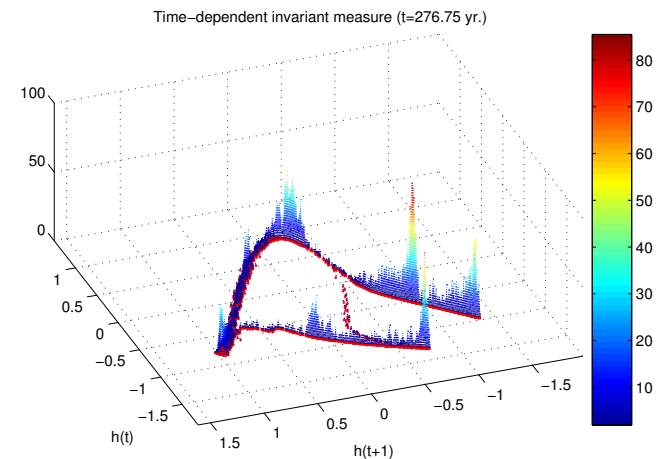


# How to define climate sensitivity or, What happens when there's natural variability?

This definition allows us to watch how “the earth moves,” as it is affected by both natural and anthropogenic forcing, in the presence of natural variability, which includes both chaotic & random behavior:



$$\gamma = \partial d_W / \partial \tau$$



# Outline

- The IPCC process: results and uncertainties
- Natural climate variability as a source of uncertainties
  - sensitivity to initial data → error growth
  - sensitivity to model formulation → see below!
- Uncertainties and how to fix them
  - structural stability and other kinds of robustness
  - non-autonomous and random dynamical systems (NDDS & RDS)
- Two illustrative examples
  - the Lorenz convection model
  - an El Niño–Southern Oscillation (ENSO) model
- Linear response theory and climate sensitivity
- **Conclusions and references**
  - natural variability and anthropogenic forcing: the “grand unification”
  - selected bibliography

# Concluding remarks, I – RDS and RAs

## *Summary*

- A change of paradigm from **closed, autonomous systems** to **open, non-autonomous ones**.
- Random attractors are (i) spectacular, (ii) useful, and (iii) just starting to be explored for climate applications.

## *Work in progress*

- Study the effect of specific **stochastic parametrizations** on model robustness.
- Applications to **intermediate models** and GCMs.
- Implications for **climate sensitivity**.
- Implications for **predictability**?

# Yet another (grand?) unification

Lorenz (*JAS*, 1963)

Climate is deterministic and autonomous,  
but highly nonlinear.

Trajectories diverge exponentially,  
forward asymptotic PDF is multimodal.

Hasselmann (*Tellus*, 1976)

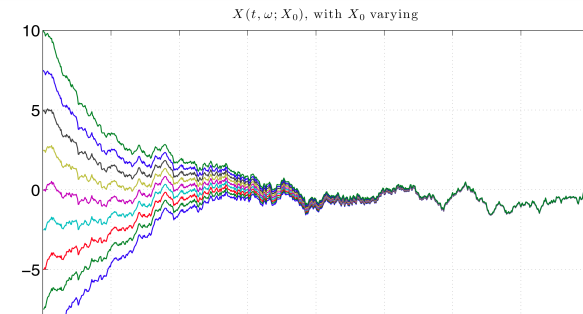
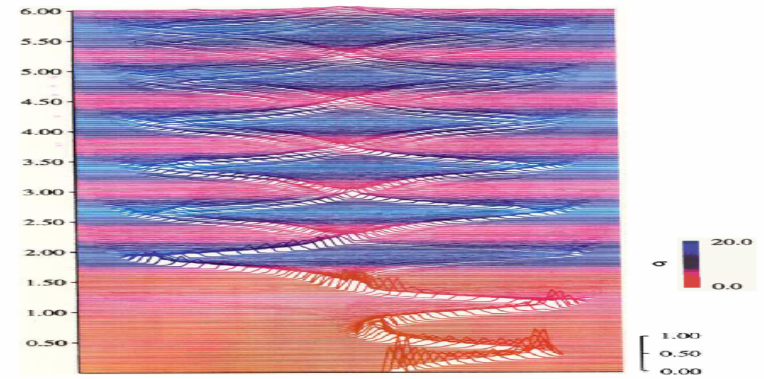
Climate is stochastic and noise-driven,  
but quite linear.

Trajectories decay back to the mean,  
forward asymptotic PDF is unimodal.

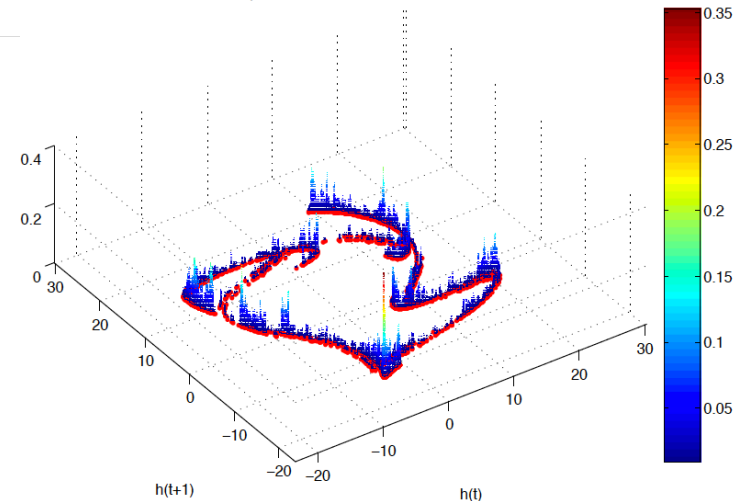
**Grand unification (?)**

Climate is deterministic + stochastic,  
as well as highly nonlinear.

Internal variability and forcing interact  
strongly, **change and sensitivity**  
refer to both mean and higher moments.



Time-dependent invariant measure of the GT-model



# Concluding remarks, II – Climate change & climate sensitivity

## *What do we know?*

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

## *What do we know less well?*

- By how much?
  - Is it getting warmer ...
  - Do we contribute to it ...
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

## *What to do?*

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity
  - stochastic structural and statistical stability!
  - linear response = response function + susceptibility function!!

# Concluding remarks, II –

## Climate change & climate sensitivity

### *What do we know?*

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

### *What do we know less well?*

- By how much?
  - Is it getting warmer ...
  - Do we contribute to it ...
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

### *What to do?*

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity
  - stochastic structural and statistical stability!
  - linear response = response function + susceptibility function!!

# Concluding remarks, II –

## Climate change & climate sensitivity

### *What do we know?*

- It's getting warmer.
- We do contribute to it.
- So we should act as best we know and can!

### *What do we know less well?*

- By how much?
  - Is it getting warmer ...
  - Do we contribute to it ...
- How does the climate system (atmosphere, ocean, ice, etc.) really work?
- How does natural variability interact with anthropogenic forcing?

### *What to do?*

- Better understand the system and its forcings.
- Explore the models', and the system's, robustness and sensitivity
  - stochastic structural and statistical stability!
  - linear response = response function + susceptibility function!!

# Some general references

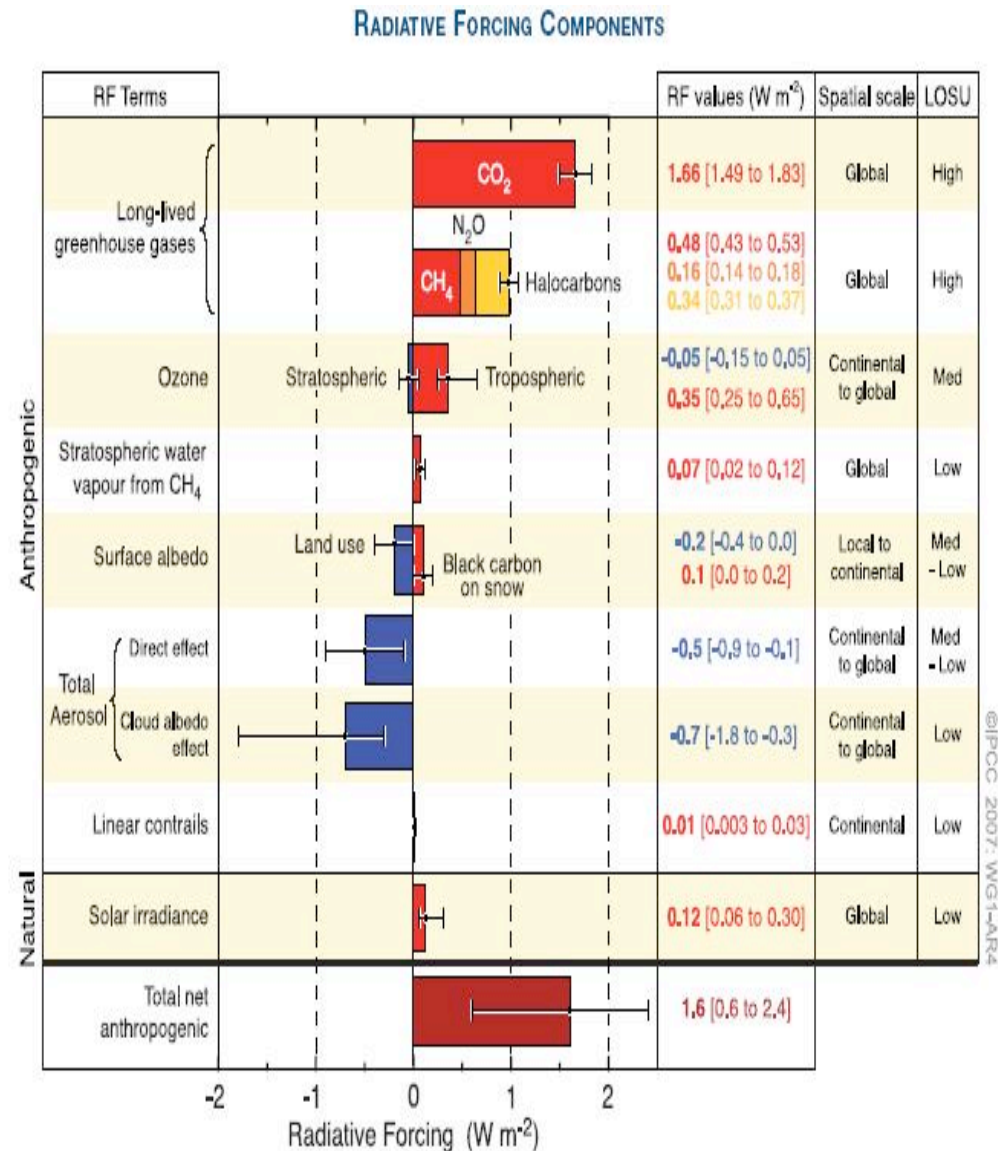
- Andronov, A.A., and L.S. Pontryagin, 1937: Systèmes grossiers. *Dokl. Akad. Nauk. SSSR*, **14**(5), 247–250.
- Arnold, L., 1998: *Random Dynamical Systems*, Springer Monographs in Math., Springer, 625 pp.
- Arnol'd, V. I., 1983: *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer-Verlag, New York/Heidelberg/Berlin, 334 pp.
- Charney, J.G., *et al.*, 1979: *Carbon Dioxide and Climate: A Scientific Assessment*. National Academic Press, Washington, D.C.
- Chekroun, M. D., E. Simonnet, and M. Ghil, 2011: Stochastic climate dynamics: Random attractors and time-dependent invariant measures, *Physica D*, **240**, 1685–1700.
- Ghil, M., R. Benzi, and G. Parisi (Eds.), 1985: *Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics*, North-Holland, 449 pp.
- Ghil, M., and S. Childress, 1987: *Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory and Climate Dynamics*, Ch. 5, Springer-Verlag, New York, 485 pp.
- Ghil, M., M.D. Chekroun, and E. Simonnet, 2008: Climate dynamics and fluid mechanics: Natural variability and related uncertainties, *Physica D*, **237**, 2111–2126.
- Houghton, J.T., G.J. Jenkins, and J.J. Ephraums (Eds.), 1991: *Climate Change, The IPCC Scientific Assessment*, Cambridge Univ. Press, Cambridge, MA, 365 pp.
- Lorenz, E.N., 1963: Deterministic nonperiodic flow. *J. Atmos. Sci.*, **20**, 130–141.
- Neelin, J.D., A. Bracco *et al.*, 2010: Considerations for parameter optimization and sensitivity in climate models, *Proc. Natl. Acad. Sci. USA*, **107**, 21,349–21,354.
- Ruelle, D., 1997: Application of hyperbolic dynamics to physics: Some problems and conjectures, *Bull. Amer. Math. Soc.*, **41**, 275–278.
- Solomon, S., *et al.* (Eds.). *Climate Change 2007: The Physical Science Basis. Contribution of Working Group I to the Fourth Assessment Report of the IPCC*, Cambridge Univ. Press, 2007.

**Reserve slides**

# GHGs rise!

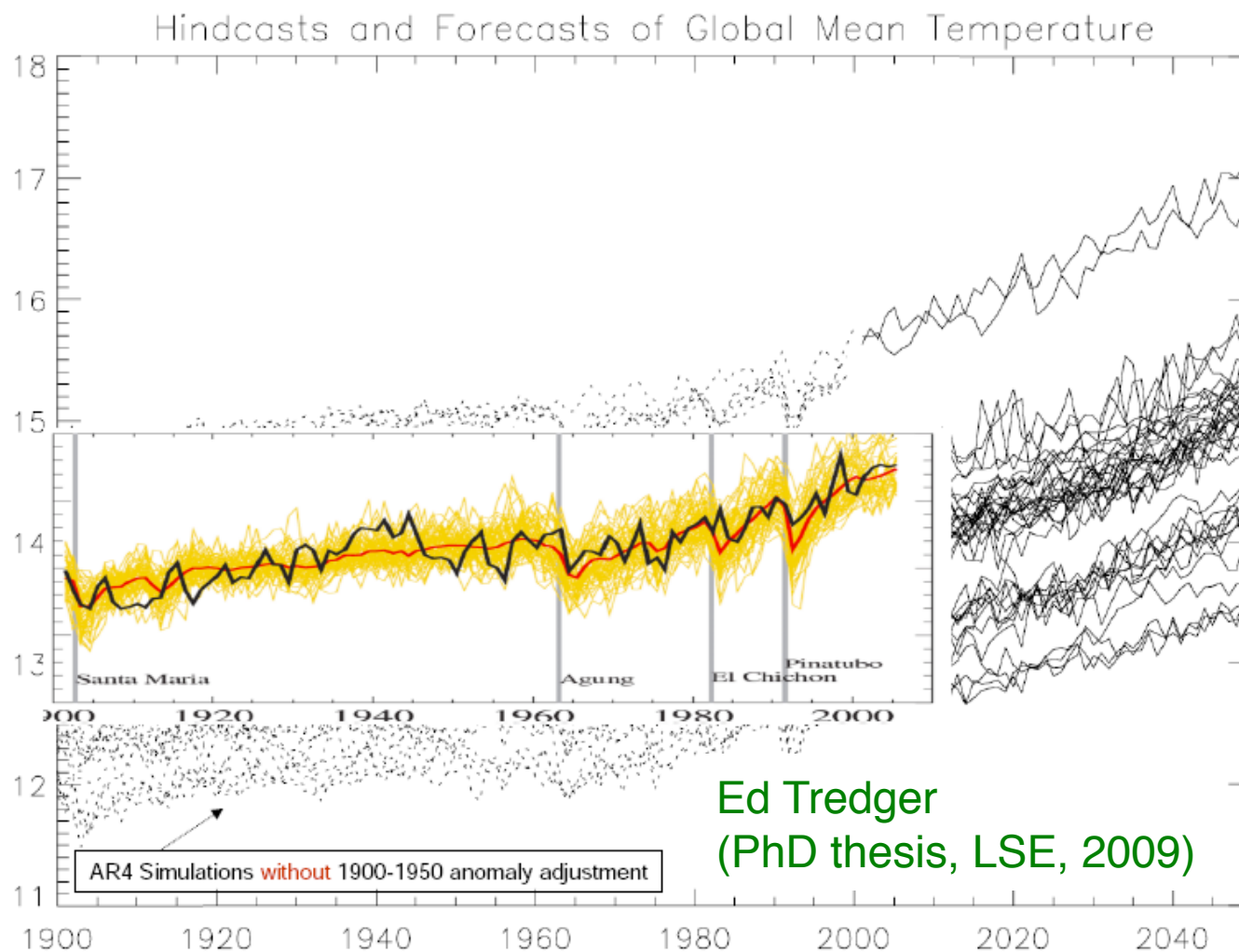
It's gotta do with us, at least a bit, ain't it?

But just how much?



IPCC (2007)

## *AR4 adjustment of 20<sup>th</sup> century simulation*



## Remarks

- We've just shown that:

$$|x(t, s; x_0) - a(t)| \xrightarrow{s \rightarrow -\infty} 0 ; \text{ for every } t \text{ fixed,}$$

and for all initial data  $x_0$ , with  $a(t) = \frac{\sigma}{\alpha}(t - 1/\alpha)$ .

- We've just encountered the concept of **pullback attraction**; here  $\{a(t)\}$  is the **pullback attractor** of the system (1).
- What does it mean physically?

The pullback attractor provides a way to assess an **asymptotic regime at time  $t$**  — the time at which we observe the system — for a system starting to evolve from the remote past  $s$ ,  $s \ll t$ .

- This asymptotic regime evolves with time: it is a dynamical object.
- Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.

## Remarks

- We've just shown that:

$$|x(t, s; x_0) - a(t)| \xrightarrow{s \rightarrow -\infty} 0 ; \text{ for every } t \text{ fixed,}$$

and for all initial data  $x_0$ , with  $a(t) = \frac{\sigma}{\alpha}(t - 1/\alpha)$ .

- We've just encountered the concept of **pullback attraction**; here  $\{a(t)\}$  is the **pullback attractor** of the system (1).
- What does it mean physically?  
The pullback attractor provides a way to assess an **asymptotic regime at time  $t$**  — the time at which we observe the system — for a system starting to evolve from the remote past  $s$ ,  $s \ll t$ .
- This asymptotic regime evolves with time: it is a dynamical object.
- Dissipation now leads to a dynamic object rather than to a static one, like the strange attractor of an autonomous system.

# Random Dynamical Systems (RDS), I - RDS theory

This theory is the counterpart for randomly forced dynamical systems (RDS) of the *geometric theory* of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space)  $\times$  (probability space).

SDE  $\sim$  ODE, RDS  $\sim$  DDS, L. Arnold (1998)  $\sim$  V.I. Arnol'd (1983).

## Setting:

- (i) A phase space  $X$ . **Example:**  $\mathbb{R}^n$ .
- (ii) A probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . **Example:** The Wiener space  $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$  with Wiener measure  $\mathbb{P}$ .
- (iii) A model of the noise  $\theta(t) : \Omega \rightarrow \Omega$  that preserves the measure  $\mathbb{P}$ , i.e.  $\theta(t)\mathbb{P} = \mathbb{P}$ ;  $\theta$  is called *the driving system*.  
**Example:**  $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)$ ;  
it starts the noise at  $s$  instead of  $t = 0$ .
- (iv) A mapping  $\varphi : \mathbb{R} \times \Omega \times X \rightarrow X$  with the cocycle property.  
**Example:** The solution operator of an SDE.

# Random Dynamical Systems (RDS), I - RDS theory

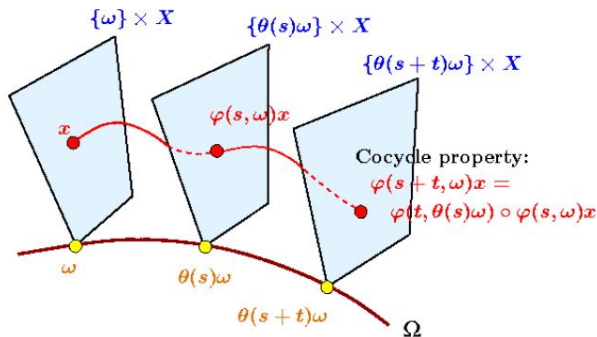
This theory is the counterpart for randomly forced dynamical systems (RDS) of the *geometric theory* of ordinary differential equations (ODEs). It allows one to treat stochastic differential equations (SDEs) — and more general systems driven by noise — as flows in (phase space)  $\times$  (probability space).

SDE  $\sim$  ODE, RDS  $\sim$  DDS, L. Arnold (1998)  $\sim$  V.I. Arnol'd (1983).

## Setting:

- (i) A phase space  $X$ . **Example:**  $\mathbb{R}^n$ .
- (ii) A probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . **Example:** The Wiener space  $\Omega = C_0(\mathbb{R}; \mathbb{R}^n)$  with Wiener measure  $\mathbb{P}$ .
- (iii) A model of the noise  $\theta(t) : \Omega \rightarrow \Omega$  that preserves the measure  $\mathbb{P}$ , i.e.  $\theta(t)\mathbb{P} = \mathbb{P}$ ;  $\theta$  is called **the driving system**.  
**Example:**  $W(t, \theta(s)\omega) = W(t + s, \omega) - W(s, \omega)$ ;  
it starts the noise at  $s$  instead of  $t = 0$ .
- (iv) A mapping  $\varphi : \mathbb{R} \times \Omega \times X \rightarrow X$  with the cocycle property.  
**Example:** The solution operator of an SDE.

# RDS, II - A Geometric View of SDEs



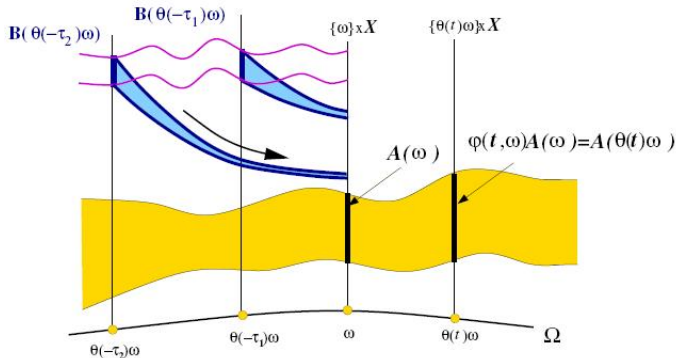
- $\varphi$  is a random dynamical system (RDS)
- $\Theta(t)(x, \omega) = (\theta(t)\omega, \varphi(t, \omega)x)$  is a flow on the bundle

# RDS, III- Random attractors (RAs)

A random attractor  $\mathcal{A}(\omega)$  is both *invariant* and “pullback” *attracting*:

- (a) **Invariant:**  $\varphi(t, \omega)\mathcal{A}(\omega) = \mathcal{A}(\theta(t)\omega)$ .
- (b) **Attracting:**  $\forall B \subset X, \lim_{t \rightarrow \infty} \text{dist}(\varphi(t, \theta(-t)\omega)B, \mathcal{A}(\omega)) = 0$  a.s.

*Pullback attraction to  $\mathcal{A}(\omega)$*



# Applications to a nonlinear stochastic El Niño model

Chekroun, Simonnet and Ghil, 2011

Timmerman & Jin (*Geophys. Res. Lett.*, 2002) have derived the following low-order, tropical-atmosphere–ocean model. The model has three variables: thermocline depth anomaly  $h$ , and SSTs  $T_1$  and  $T_2$  in the western and eastern basin.

$$\begin{aligned}\dot{T}_1 &= -\alpha(T_1 - T_r) - \frac{2\varepsilon u}{L}(T_2 - T_1), \\ \dot{T}_2 &= -\alpha(T_2 - T_r) - \frac{w}{H_m}(T_2 - T_{sub}), \\ \dot{h} &= r(-h - bL\tau/2).\end{aligned}$$

The related diagnostic equations are:

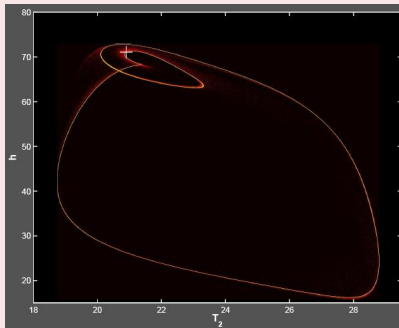
$$\begin{aligned}T_{sub} &= T_r - \frac{T_r - T_{r0}}{2} [1 - \tanh(H + h_2 - z_0)/h^*] \\ \tau &= \frac{a}{\beta} (T_1 - T_2) [\xi_t - 1].\end{aligned}$$

- $\tau$ : the wind stress anomalies,  $w = -\beta\tau/H_m$ : the equatorial upwelling.
- $u = \beta L\tau/2$ : the zonal advection,  $T_{sub}$ : the subsurface temperature.

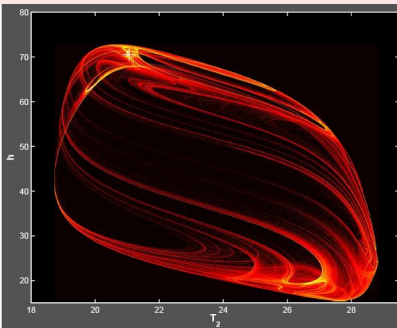
**Wind stress bursts** are modeled as white noise  $\xi_t$  of variance  $\sigma$ , and  $\varepsilon$  measures the strength of the **zonal advection**.

# Random attractors: the frozen statistics

## Random Shil'nikov horseshoes



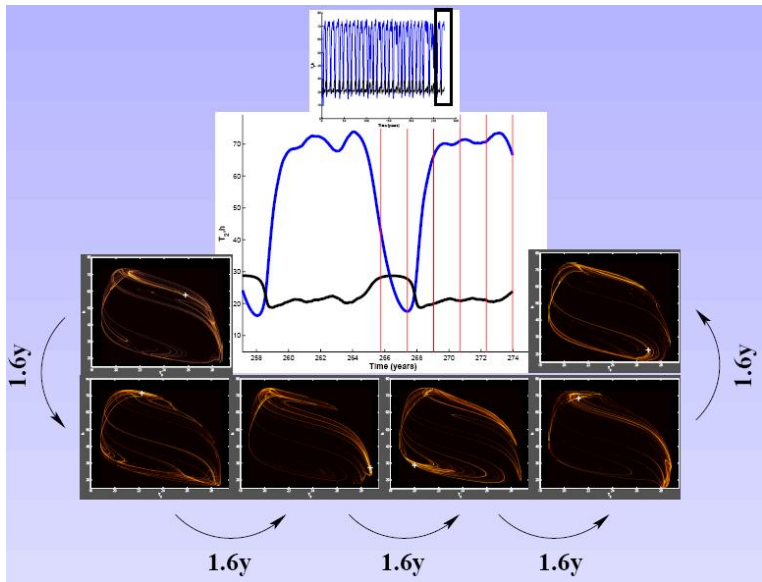
$$\sigma=0.005$$



$$\sigma=0.05$$

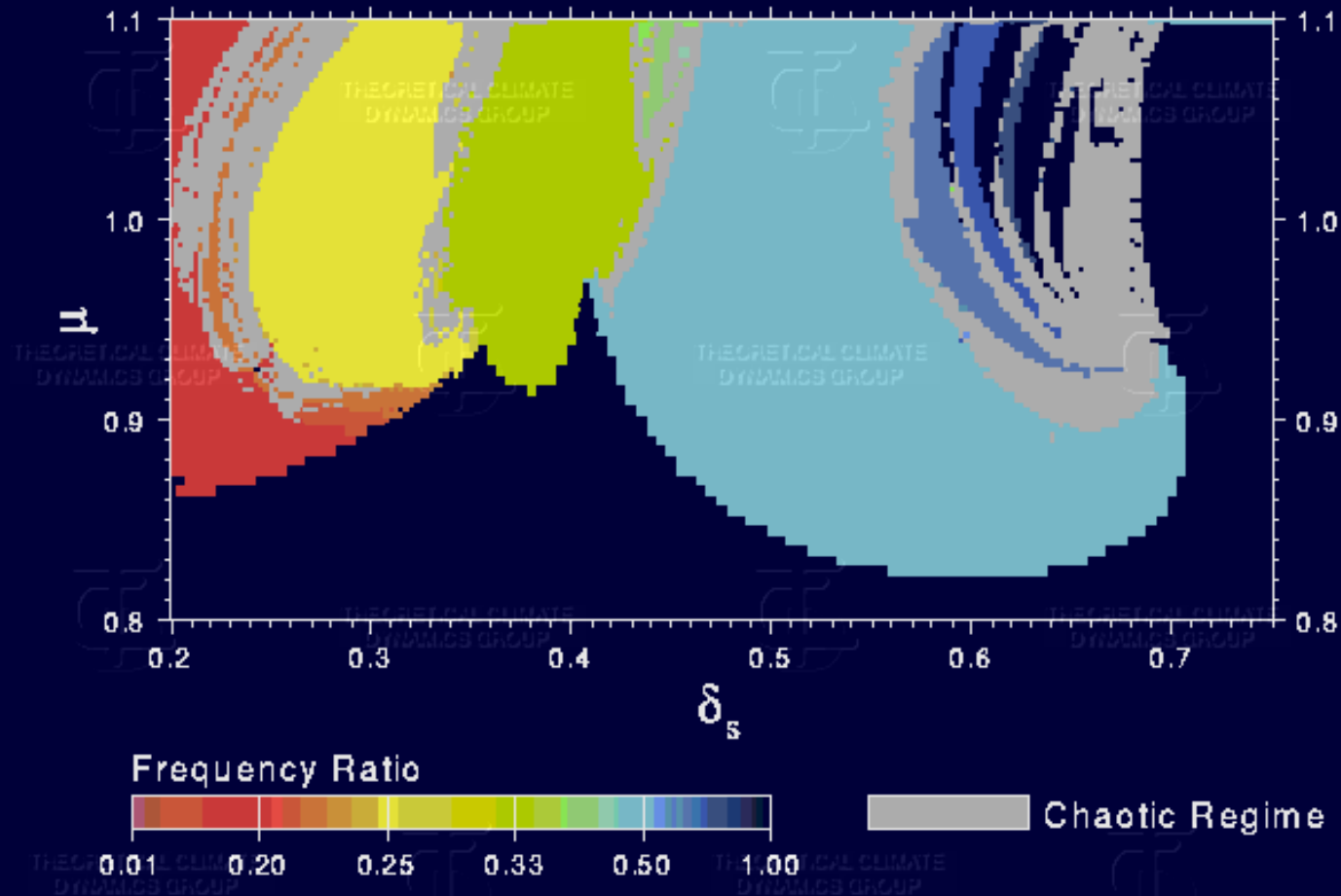
- Horseshoes can be noise-excited,  
left: a weakly-perturbed limit cycle, right: the same with larger noise.
- Golden: most frequently-visited areas; white 'plus' sign: most visited.

# An episode in the random's attractor life



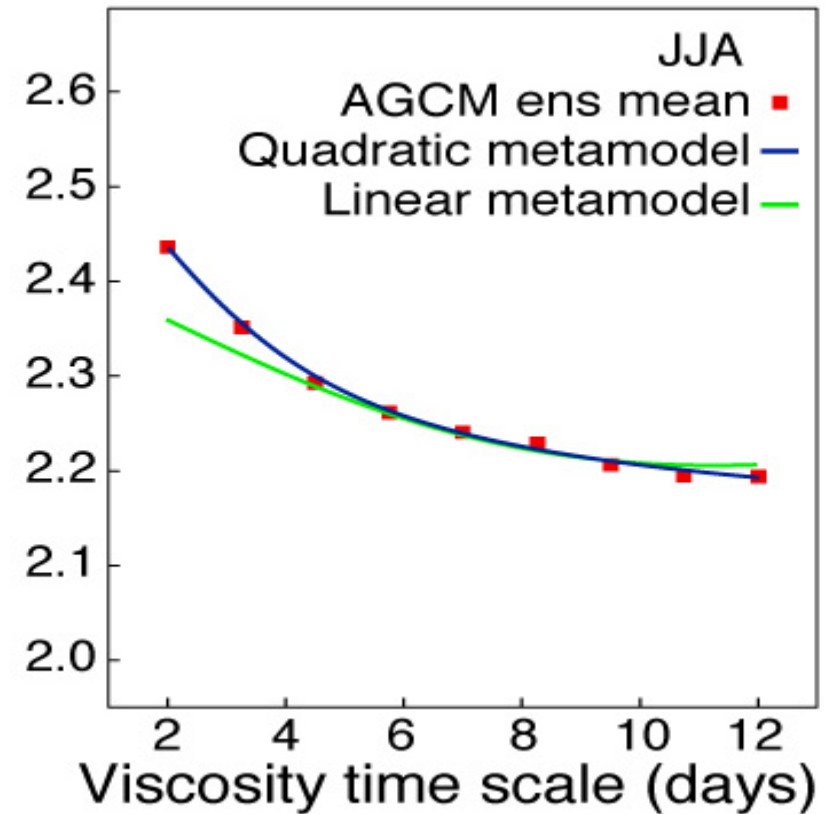
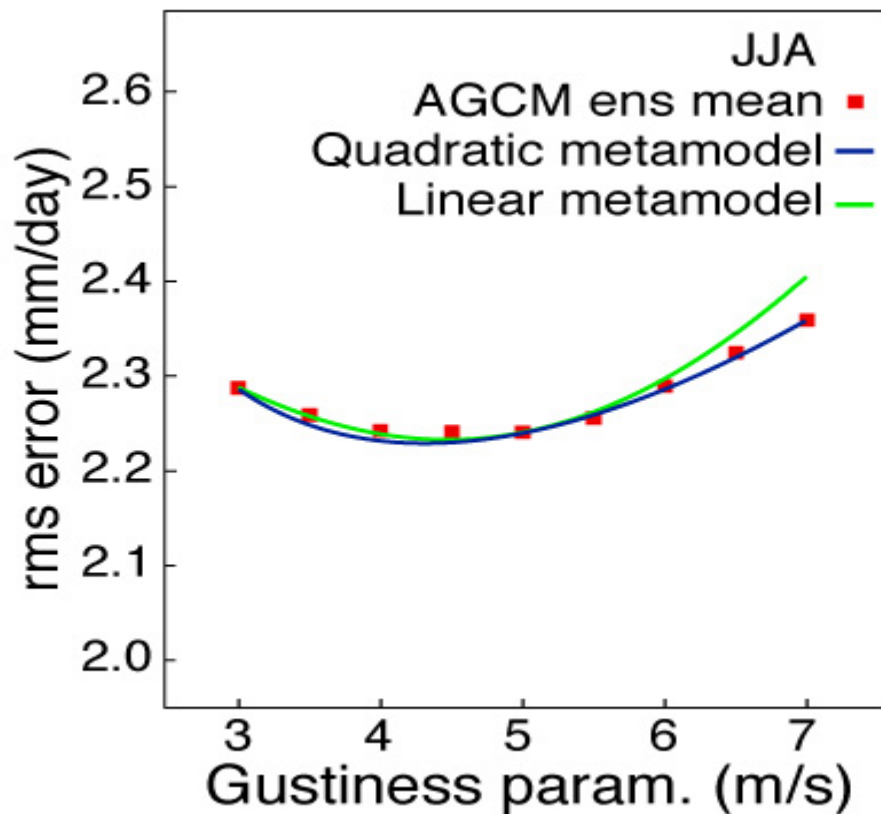
# Devil's Bleachers in a 1-D ENSO Model

Ratio of ENSO frequency to annual cycle



# Parameter dependence – II

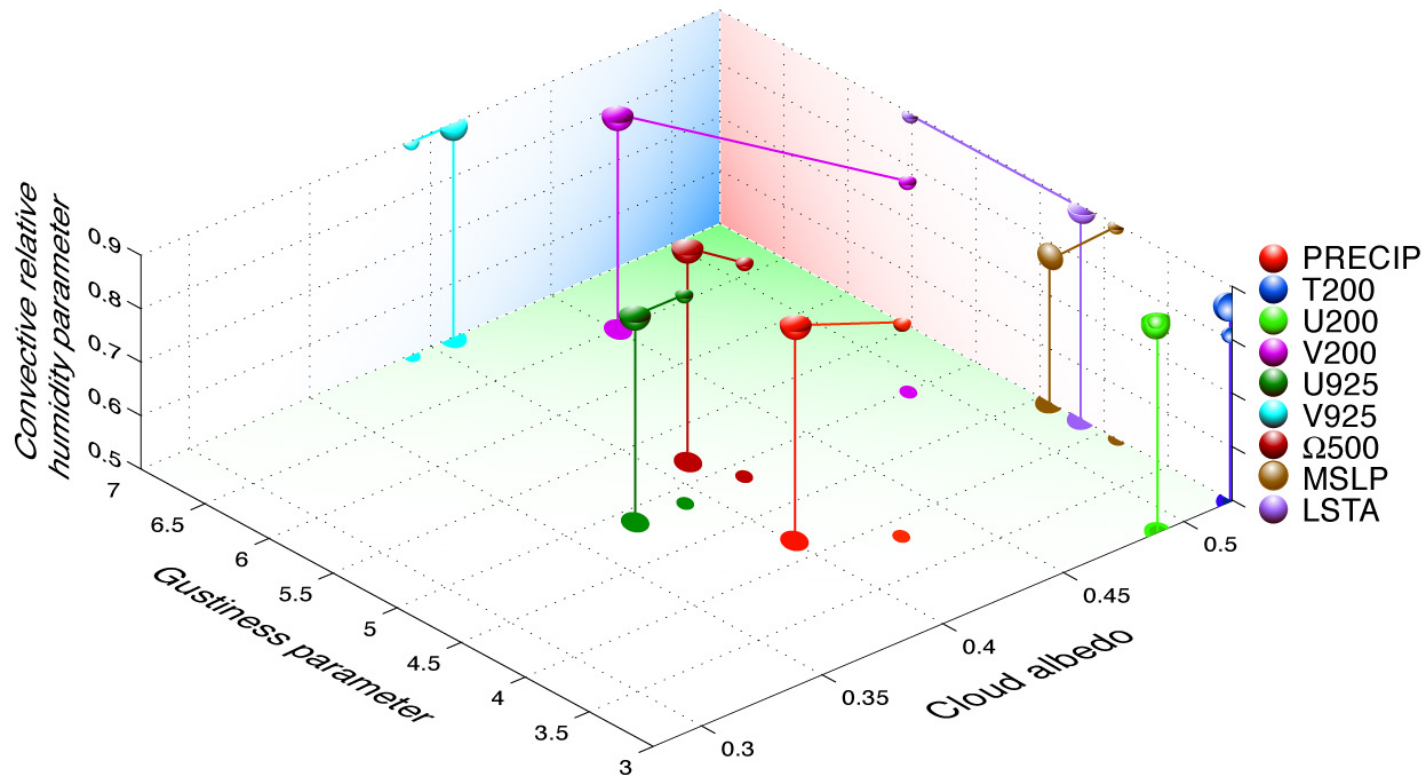
When it is smooth, one can optimize a GCM's single-parameter dependence



ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)

# Parameter dependence – III

Multi-objective algorithms avoid arbitrary weighting of criteria in a unique cost function:

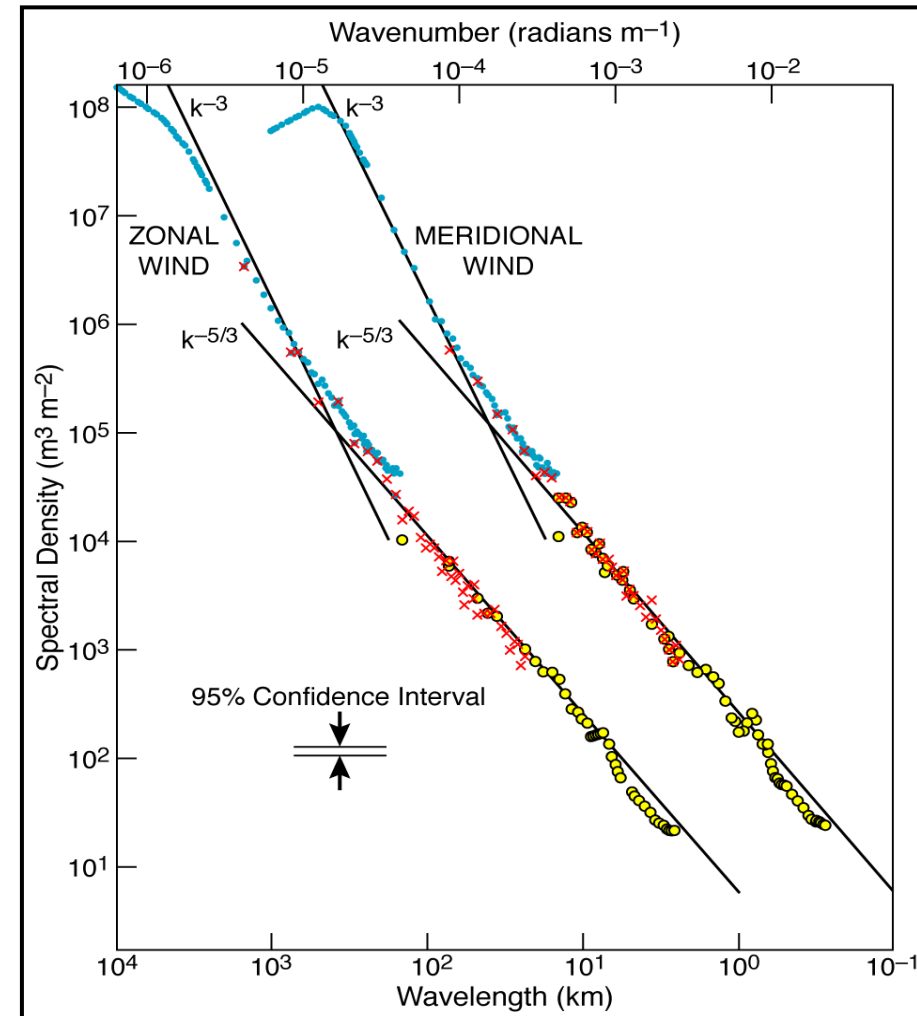


Optimization algorithms that are  $\mathcal{O}(N)$  and  $\mathcal{O}(N^2)$ , rather than  $\mathcal{O}(S^N)$ , where  $N$  is the number of parameters and  $S$  is the sampling density.

ICTP AGCM (Neelin, Bracco, Luo, McWilliams & Meyerson, *PNAS*, 2010)

# But deterministic chaos doesn't explain all: there are many other sources of irregularity!

- The energy spectrum of the atmosphere and ocean is “full”: all space & time scales are active and they all contribute to forecasting uncertainties.
- Still, one can imagine that the longest & slowest scales contribute most to the longest-term forecasts.
- “One person’s signal is another person’s noise.”



After Nastrom & Gage (*JAS*, 1985)

# Climatic uncertainties & moral dilemmas



**Thought leaders**  
Rice, top left, spoke of multilateralism, while Bono, left, demanded more action on poverty. Presidents Karzai and Musharraf, right, both face troubles at home

♥ ... keep today's climate for tomorrow?



**Agitator Gore**  
wants a global compact to tackle climate change and poverty

♥ **Feed the world today or...**

Davos, Feb. 2008, photos by *TIME Magazine*, 11 Feb. '08;  
see also Hillerbrand & Ghil, *Physica D*, 2008, **237**, 2132–2138,  
[doi:10.1016/j.physd.2008.02.015](https://doi.org/10.1016/j.physd.2008.02.015) .